THE

FIRST BOOK OF EUCLID

MADE EASY FOR BEGINNERS

ARRANGED FROM 'THE ELEMENTS OF EUCLID' BY ROBERT SIMSON M D

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ADVERTISEMENT.

THE IDEA of this little book was suggested to the Publishers by a gentleman whose son was at one of our great public schools On examination he found that though his son was supposed to have learned the first two books of Euclid, he did not really understand the first Proposition

The first five Propositions were thereupon written out by the father exactly as they appear in this volume, and the son not only easily mastered them, but had little subsequent difficulty with his Euclid It is hoped that what was found useful to one boy will be found useful to many others

It may be pointed out that this book is in no sense a crib It aims at enabling a beginner to understand the Problems of Euclid by avoiding the difficulty to many youthful minds occasioned by the use of letters, while it also removes the temptation to endeavour to repeat the Problems by rote

Only the first book of Euclid is given, for it is believed that if the first book be thoroughly understood by the student, the other books will offer little difficulty to him. Indeed, if a youth, after mastering the first book, cannot understand the other books, it is useless for him to continue his study of Euclid.

THE

ELEMENTS OF EUCLID.

BOOK I.

DEFINITIONS

I

A point is that which has no parts and no size

II

A line is length without breadth

III

The ends of a line are points

IV

A straight line is that which lies evenly between its farthest points

V

A superficies is that which has only length and breadth

VI

The ends of a superficies are lines.

A plane superficies is that in which, if any two points be taken, the straight line between them lies entirely in that superficies.

VIII

A plane angle is formed by the meeting of two lines which are in the same plane but are not in the same direction.

IX

A plane rectilineal (straight line) angle is formed by the meeting of two straight lines, which are not in the same straight line.

N.B. When there are more angles than one at one point, any one of them is expressed by the lines forming the angle; thus, the angles in the diagram are respectively the angle at the point where the red line meets the black line, and the angle at the point where the blue line meets the black line.

X

When one straight line standing on another straight line makes the adjacent (adjoining) angles equal to one another, each of the angles is called a right angle, and the straight line which stands on the other is called a perpendicular to it.

XI

An obtuse angle is that which is greater than a right angle.

XII

An acute angle is that which is less than a right angle

XIII

A term or boundary is the end of any thing

XIV

A figure is that which is inclosed by one or more boundaries

XV

A circle is a plane figure contained by one line, which is called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another

XVI

And this point is called the centre of the circle

XVII

A diameter of a circle is a straight line drawn through the centre and terminated both ways by the circumference

XVIII

A semicircle (i e half circle) is the figure contained by a diameter and the part of the circumference cut off by the diameter

XIX

A segment of a circle is the figure contained by line, and the circumference it cuts off

XX

Rectilineal figures are those which are con straight lines

XXI

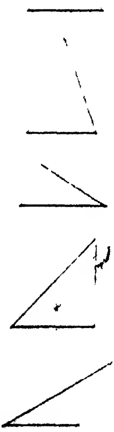
Trilateral (i.e. three-sided) figures, or triangles, are figures contained by three straight lines.

XXII

Quadrilateral (i.e. four sided) figures are figures contained by four straight lines.

XXIII

Multilateral (i.e. many sided) figures, or polygons, are figures contained by more than four straight lines.



XXIV

Of three sided figures, an equilateral triangle is that which has three equal sides.

XXV

An isosceles triangle is that which has only two sides equal.

XXVI

A scalene triangle is that which has three unequal sides.

XXVII

A right-angled triangle is that which has a right angle.

XXVIII

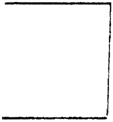
An obtuse-angled triangle is that which has an obtuse angle.

XXIX

An acute-angled triangle is that which has three acute angles

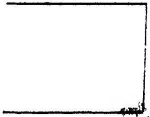
XXX

Of four-sided figures, a square is that which has all its sides equal and all its angles right angles



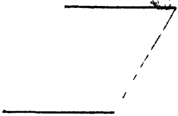
XXXI

An oblong is that which has all its angles right angles, but has not all its sides equal



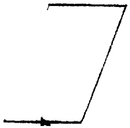
XXXII

A rhombus is that which has all its sides equal, but its angles are not right angles



XXXIII

A rhomboid is that which has its opposite sides equal to one another, but all its sides are not equal, and its angles are not right angles

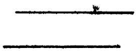


XXXIV

All other four-sided figures besides these are called trapeziums.

XXXV

Parallel straight lines are lines in the same plane, which, however much they may be lengthened both ways, do not meet



POSTULATES

(i e *Positions supposed or assumed without proof*)

I

Let it be granted that a straight line may be drawn from any one point to any other point

II

That a terminated straight line may be lengthened to any distance in a straight line

III

And that a circle may be made from any centre, at any distance from that centre

AXIOMS

(i e *Self-evident truths*)

I

Things which are equal to the same thing are equal to one another

II

If equals be added to equals the wholes are equal

III

If equals be taken from equals the remainders are equal

IV

If equals be added to unequals the wholes are unequal

V

If equals be taken from unequals the remainders are unequal

VI

Things which are double of the same are equal to one another

VII

Things which are halves of the same are equal to one another

VIII

Magnitudes which coincide with one another, i e which exactly fill the same space, are equal to one another

IX

The whole is greater than its part

X

Two straight lines cannot inclose a space

XI

All right angles are equal to one another

XII

If a straight line meets two straight lines so as to make the two inside angles on the same side of it taken together less than two right angles, these straight lines if continually lengthened shall finally meet upon that side on which are the angles which are less than two right angles

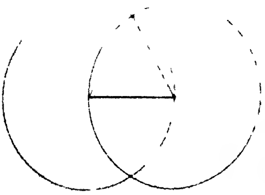
PROPOSITION I.—PROBLEM

To describe (make) an equilateral (equal-sided) triangle upon a given finite straight line.

First, we draw a straight line on which to make an equilateral (equal sided) triangle.

Second, we make a circle, taking one end of the line as the centre, and the other end of the line as the distance, as is allowed by Postulate III., which says that a circle may be made from any centre at any distance from that centre.

Third, we make another circle, taking the other end of the line as the centre, and the end of the line which we before took as the centre, as the distance.



Fourth, from either of the points at which the circles cut each other, we draw two straight lines to the ends of the line on which we have to make the equilateral (equal-sided) triangle, as is allowed by Postulate I., which says that a straight line may be drawn from any one point to any other point.

Then the black, blue, and red lines form an equilateral (equal-sided) triangle, and we have to prove that they do so.

Because one end of the black line is the centre of the red circle, the red line is equal to the black line; for, according to Definition XV., all straight lines drawn from a certain point within a circle—namely, the centre of the circle—to its circumference (the line round the circle) are equal to one another. And because the other end of the black line is the centre of the blue circle, the blue line is equal to the black line for the same reason. And it has been proved that the red line is equal to the black line; therefore the blue line and the red line are each of them equal to the black line; and, according to Axiom I., things which are equal to the same thing are equal to one another; therefore the blue line is equal to the red line. Consequently the blue line, the red line, and the black line are equal to one another, and the triangle formed by the three lines is an equilateral (equal-sided) triangle; and it has been made on the black line, which was the given straight line; so that what was required has been done.

PROPOSITION II.—PROBLEM.

From a given point to draw a straight line equal to a given straight line.

First, let us draw the line which is the given straight line, and mark the point from which we are to draw a line equal to this line.

Second, as allowed by Postulate I., which says that a

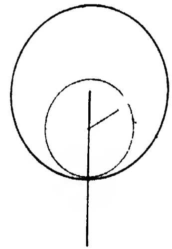
straight line may be drawn from any one point to any other point, let us draw a line from the point to the end of the line

Third, upon the line thus drawn let us make an equilateral triangle, as shown in Proposition I

Fourth, as allowed by Postulate II, which says that a terminated straight line may be lengthened to any distance in a straight line, we lengthen the black side and the yellow side of the triangle thus made

Fifth, from the end of the blue line as centre at a distance equal to its length we make a circle as allowed by Postulate III, which says that a circle may be made from any centre, at any distance from that centre

Sixth, from the point where the black side and the yellow side of the triangle meet, as centre, at a distance equal to the length of the black line to where it is cut by the red circle, we make the black circle



The yellow line from the red line to the black circle is equal to the blue line, and we have to prove it to be so

Because one end of the blue line is the centre of the red circle, the blue line is equal to the black line from that centre to the red circle, according to Definition XV, which says that all lines drawn from the centre of a circle to its circumference are equal And because the point where the black line and the yellow line meet is the centre of the black

circle, the black line from that point to the black circle is equal to the yellow line from that point to the black circle. But we know that so much of these lines as form the sides of the equilateral triangle are equal; therefore, according to Axiom III, which says that if equals be taken from equals the remainders are equal, the remainders of the two lines are equal. But we have shown that this remainder of the black line is equal to the blue line, therefore the blue line and the remainder of the yellow line are each equal to the remainder of the black line. And we know by Axiom I that things which are equal to the same thing are equal to one another, therefore the blue line and the remainder of the yellow line are equal to one another. And this remainder of the yellow line is drawn from the given point: which was required to be done.

Proposition III.—Problem

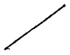
From the greater of two given straight lines to cut off a part equal to the less.

First, we draw two lines, one of which is greater than the other.

Second, from either end of the red line we draw, as shown in Proposition II, a line equal to the blue line.

Third, from the point where the black line meets the red line as centre, with the other end of the black line as

distance, we make a circle, as allowed by Postulate III, which says that a circle may be made from any centre at any distance from that centre



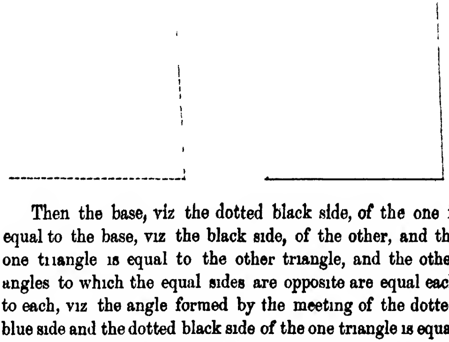
The portion of the red line from the black line to the yellow circle is equal to the blue line, and we have to prove it

Because the end of the black line is the centre of the yellow circle, the black line is equal to the red line from the centre to the circumference of the circle, according to Definition XV, which says that all lines drawn from the centre of a circle to its circumference are equal to one another But the blue line is equal to the black line because it was made so Therefore the blue line, and the red line from the centre to the circumference of the yellow circle, are each of them equal to the black line Consequently, according to Axiom I, which says that things which are equal to the same thing are equal to one another, the red line from the centre to the circumference of the yellow circle is equal to the blue line And it is a portion cut off from the given red line which was required to be done

Proposition IV.—Theorem

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by those sides equal to one another, they shall likewise have their bases, or third sides, equal, and the two triangles shall be equal, and their other angles shall be equal each to each, viz. those to which the equal sides are opposite.

First, we draw two triangles, having the dotted blue side of the one equal to the blue side of the other, and the dotted red side of the one equal to the red side of the other, and the angle formed by the dotted blue side and the dotted red side of the one equal to the angle formed by the blue side and the red side of the other.



Then the base, viz the dotted black side, of the one is equal to the base, viz the black side, of the other, and the one triangle is equal to the other triangle, and the other angles to which the equal sides are opposite are equal each to each, viz the angle formed by the meeting of the dotted blue side and the dotted black side of the one triangle is equal to the angle formed by the meeting of the blue side and the black side of the other triangle, and the angle formed by the meeting of the dotted red side and the dotted black side in the one triangle is equal to the angle formed by the meeting

of the red side and the black side of the other triangle And all this we have to prove

Let us place the one triangle upon the other triangle so that the point where the dotted blue side and the dotted red side of the one triangle meet may be on the point where the blue side and the red side of the other triangle meet, and the dotted blue side of the one upon the blue side of the other, then the other end of the dotted blue side shall coincide with the other end of the blue side, for we know by the Hypothesis that the two sides are equal And the dotted blue side coinciding with the blue side, the dotted red side shall also coincide with the red side, for by the Hypothesis the angle formed by the meeting of the dotted blue side and the dotted red side in the one triangle is equal to the angle formed by the meeting of the blue side and the red side in the other triangle And the other end of the dotted red side shall coincide with the other end of the red side, because by the Hypothesis these two sides are equal And the other end of the dotted blue side was proved to coincide with the other end of the blue side, therefore the base, viz the dotted black side, of the one triangle shall coincide with the base, viz the black side, of the other triangle, because otherwise two straight lines would inclose a space, which is impossible according to Axiom X, which says two straight lines cannot inclose a space. And the two bases therefore coincide, they are also equal according to Axiom VIII, which says that magnitudes which coincide with one another, i e which exactly fill the same space, are equal to one another Consequently, the one triangle coincides with the other triangle and is equal to it, and the other angles of the one coincide with the remaining angles of the other, and are equal to them, viz. the angle formed by the meeting of the dotted black side and the dotted blue side of the one triangle equal to the angle formed by the meeting of the black side and the blue side of the other triangle, and the angle formed by the

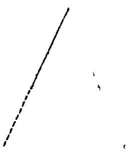
meeting of the dotted black side and the dotted red side of the one triangle equal to the angle formed by the meeting of the black side and the red side of the other triangle And these are the angles opposite the equal sides Therefore, if two triangles have two sides of the one equal to two sides of the other each to each, and have likewise the angles contained by these sides equal to one another, their bases shall likewise be equal, and the triangles shall be equal, and their other angles to which the equal sides are opposite shall be equal each to each which is what we had to show

Proposition V —Theorem

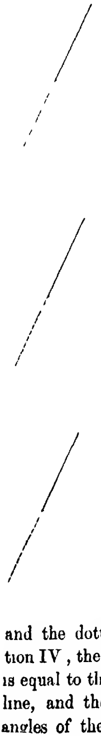
The angles at the base of an isosceles triangle (i e a triangle which has only two sides equal) are equal to one another, and if the equal sides be produced (lengthened), the angles upon the other side of the base shall be equal

First, we draw an isosceles triangle (i e a triangle which has only two sides equal), and make the black side equal to the red side

Second, we produce (lengthen) the two equal sides, then the angle made by the meeting of the black side and the blue side of the triangle shall be equal to the angle made by the meeting of the red side and the blue side of the triangle, and the angle made by the meeting of the dotted black line and the blue side of the triangle shall be equal to the angle made by the meeting of the dotted red line and the blue side of the triangle This we have to prove



Third, in the dotted black line take any point, and from the dotted red line cut off, according to Proposition III., a part equal to the part taken of the dotted black line. Join the point taken in the dotted black line with the end of the blue line, and the corresponding point found in the dotted red line with the other end of the blue line.



Then because the line composed of the black line and the part taken of the dotted black line is equal to the line composed of the red line and the part found in the dotted red line (by the Construction), and the black line is equal to the red line according to the Hypothesis, the two sides, viz. the line composed of the black side and the part of the dotted black line, and the red side, are equal to the two sides, viz. the line composed of the red side and the part of the dotted red line, and the black side, each to each, and they contain the angle made by the meeting of the black side and the red side belonging to each of the two triangles, the one of which is formed by the black side and the part of the dotted black line, the red side and the yellow line, and the other of which is formed by the red side and the part of the dotted red line, the black side and the dotted blue line, therefore, according to Proposition IV., the base of the one triangle, viz. the yellow line, is equal to the base of the other triangle, viz. the dotted blue line, and the two triangles are equal, and the remaining angles of the one are equal to the remaining angles of the

other, each to each, to which the equal sides are opposite, viz the angle formed by the meeting of the red side and the yellow side to the angle formed by the meeting of the black side and the dotted blue side, and the angle formed by the meeting of the part taken of the dotted black line and the yellow side, to the angle formed by the meeting of the corresponding part found in the dotted red line, and the dotted blue side Again, because the black line and the part of the dotted black line are together equal to the red line and the part of the dotted red line, and parts of these, viz the black line and the red line, are equal by the Hypothesis, therefore, according to Axiom III, which says that if equals be taken from equals the remainders are equal, the remainder, viz the part of the dotted black line, is equal to the remainder, viz the part of the dotted red line And the yellow line was proved equal to the dotted blue line, therefore the two sides, the part of the dotted black line and the yellow line, are equal to the two sides, the part of the dotted red line and the dotted blue line, each to each And the angle at the point where the part taken of the dotted black line meets the yellow line was proved equal to the angle at the point where the part found in the dotted red line meets the dotted blue line And the base, the blue line, is part of each of the two triangles, the one of which is formed by the part of the dotted black line, the yellow line, and the blue line, and the other by the part of the dotted red line, the dotted blue line, and the blue line, therefore, according to Proposition IV, these two triangles are equal, and their remaining angles are equal each to each to which the equal sides are opposite, therefore the angle formed by the meeting of the dotted black line and the blue line is equal to the angle formed by the meeting of the dotted red line and the blue line, and the angle formed by the meeting of the blue line and the yellow line equal to the angle formed by the meeting of the blue line and the dotted blue line And as

we have shown that the large angle formed by the meeting of the black side and the dotted blue line is equal to the large angle formed by the meeting of the red side and the yellow line, and that parts of these angles, viz the small angle formed by the meeting of the blue line and the dotted blue line, is equal to the small angle formed by the meeting of the blue line and the yellow line, therefore, according to Axiom III, which says that if equals be taken from equals the remainders are equal, the remainder, viz the angle formed by the meeting of the black side and the blue side, is equal to the remainder, viz the angle formed by the meeting of the red side and the blue side And these are the angles at the base of the isosceles triangle (i e a triangle having only two sides equal) which we had to prove equal And we have already proved that the angle formed by the meeting of the dotted black line and the blue line is equal to the angle formed by the meeting of the dotted red line and the blue line, and these are the angles on the other side of the base which we had also to prove were equal so that we have proved what was required

Corollary (i e another fact proved) From the above it follows that every equilateral (equal-sided) triangle is also equiangular (i e has all its angles equal)

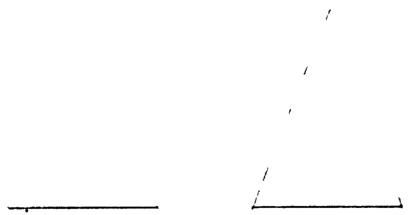
Proposition VI —Theorem

If two angles of a triangle be equal to one another, the sides also which subtend or are opposite to the equal angles shall be equal to one another

First, we make a triangle in which the angle formed by the meeting of the red side and the black side is equal to the angle formed by the meeting of the blue side and the black side We have to prove that the blue side is equal to the red side

If the blue side be not equal to the red side, one of them is greater than the other. Let us suppose the red side to be the greater, and from it cut off, as shown in Proposition III., a part equal to the blue line. Join the end of the part cut off from the red line with the end of the black line.

Then, because in the triangles the one of which is formed by the part cut off the red side, the black side, and the yellow side, and the other of which is formed by the blue side, the black side, and the long red side, the part of the red side is equal to the blue side, and the black side is part of each triangle, therefore the two sides, viz. the part cut off the red side, and the black side, are equal to the two sides, viz. the blue side and the black side, each to each, and the angle formed by the meeting of the red side and the black



side is, we know by the Hypothesis, equal to the angle formed by the meeting of the blue side and the black side, therefore the base, viz. the yellow line, is equal to the base, viz. the long red side, and the triangle formed by the part cut off the red side, the black side, and the yellow side, is equal, according to Proposition IV., to the triangle formed by the blue side, the black side, and the long red side. That is to say, the less is equal to the greater, which is absurd. Therefore the long red side is not unequal to the blue side, and therefore it must be equal to it: which is what we had to prove.

Corollary (i e another fact proved) —From the above it follows that every equiangular triangle (i e a triangle having all its angles equal) is also equilateral (i e equal-sided)

PROPOSITION VII —THEOREM

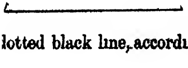
Upon the same base, and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and likewise those which are terminated in the other extremity

First, we draw a line to form the base

Second, if it be possible let us draw upon this base and upon the same side of it two triangles, which have their blue and red sides ending at one extremity of the base equal to one another, and their yellow and dotted blue sides ending at the other extremity of the base equal to one another

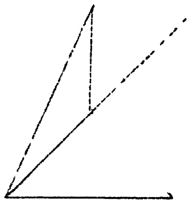
Third, join the vertices (i e top points) of the two triangles

Then in the case in which the vertex (i e top point) of each triangle is outside the other triangle, because the blue line is equal to the red line (according to the Hypothesis), the angle formed by the meeting of the blue line and the dotted black line is equal to the angle formed by the meeting of the red line and the dotted black line, according to Proposition V. But the angle



formed by the meeting of the blue line and the dotted black line is greater than the angle formed by the meeting of the yellow line and the dotted black line, according to Axiom IX, which says that the whole is greater than its part Therefore the angle formed by the meeting of the red line and the dotted black line is greater than the angle formed by the meeting of the yellow line and the dotted black line Much more then is the angle formed by the meeting of the dotted blue line and the dotted black line greater than the angle formed by the meeting of the yellow line and the dotted black line Again, because the yellow line is equal to the dotted blue line, according to the Hypothesis, it follows by Proposition V that the angle formed by the meeting of the dotted blue line and the dotted black line is equal to the angle formed by the meeting of the yellow line and the dotted black line But it has been shown to be greater than it, which is absurd, as it cannot be both greater and less than it

Now let us make the triangles in a different position, so that the vertex (i e top point) of the one shall be inside the vertex (i e top point) of the other



Produce (i e lengthen) the blue and the red sides Then because the blue side is equal to the red side (according to the Hypothesis) in the triangle formed by the blue side, the dotted black side, and the red side, it follows by Proposition V that the angles on the other side of the base (i e of the dotted black line) are equal to one another, i e the angle formed by the thin blue line and the dotted black line is equal to the angle formed by the dotted red line and the

dotted black line But the angle formed by the thin blue line and the dotted black line is greater than the angle formed by the yellow line and the dotted black line, according to Axiom IX, which says the whole is greater than its part, wherefore the angle formed by the dotted red line and the dotted black line is also greater than the angle formed by the yellow line and the dotted black line Much more then is the angle formed by the dotted blue line and the dotted black line greater than the angle formed by the yellow line and the dotted black line Again, because the yellow line is equal to the dotted blue line according to the Hypothesis, it follows by Proposition V that the angle formed by the dotted blue line and the dotted black line is equal to the angle formed by the yellow line and the dotted black line But the angle formed by the dotted blue line and the dotted black line has been proved to be greater than the angle formed by the yellow line and the dotted black line, which is impossible, for it cannot be both greater and less than it

The case in which the vertex (i e top point) of one triangle is upon the side of the other needs no proof, because it is plainly impossible

And these are the only three positions in which the triangles could be, therefore we have shown what we wanted to prove, viz that upon the same base, and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and likewise those which are terminated in the other extremity

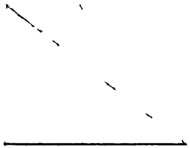
Proposition VIII.—Theorem

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides equal to them, of the other.

First, we make two triangles having two sides of the one equal to two sides of the other, each to each, viz. the dotted blue side equal to the blue side, and the dotted red side equal to the red side; and also the base, viz. the dotted black side, equal to the black side. The angle formed by the meeting of the dotted blue side and the dotted red side shall be equal to the angle formed by the meeting of the blue side and the red side. This we have to prove.

If the dotted triangle be placed upon the other triangle so that the point where the dotted blue side and the dotted black side meet be upon the point where the blue side and the black side meet, and the straight dotted black line upon the straight black line, the other end of the dotted black line shall coincide with (i.e. exactly fill the same space as) the other end of the black line, because by the Hypothesis the dotted black line is equal to the black line. Therefore the dotted black line coinciding with the black line, the dotted blue line, and the dotted red line, shall coincide with

the blue line and the red line For if the base, viz the dotted black line, coincides with the base, viz the black line, but the sides, viz the dotted blue line and the dotted red line, do not coincide with the sides, viz the blue line and the red line, but have a different position, then upon the same



base, viz the black line, and upon the same side of it there can be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and likewise their sides which are terminated in the other extremity equal But we have shown in Proposition VII that this is impossible therefore if the base, viz the dotted black line, coincides with the base, viz the black line, the sides, viz the dotted blue line and the dotted red line, must coincide with the sides viz the blue line and the red line Wherefore also the angle formed by the meeting of the dotted blue line and the dotted red line coincides with the angle formed by the meeting of the blue line and the red line, and is therefore equal to it, according to Axiom VIII, which says that magnitudes which coincide with one another are equal to one another which is what we had to prove

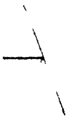
PROPOSITION IX.—PROBLEM

To bisect a given rectilineal angle (i e an angle contained by straight lines), that is, to divide it into two equal angles

First, we make the angle which it is required to bisect

Second, we take any point in the blue line, and from the red line cut off according to Proposition III a part equal to the part taken in the blue line, and join the ends of the two parts thus taken of the blue line and the red line

Third, upon the black line we make an equilateral triangle as shown in Proposition I



Fourth, we join the point where the blue line and the red line meet with the point where the dotted blue line and the dotted red line meet The straight yellow line shall bisect (i e divide into halves) the angle formed by the meeting of the blue line and the red line, and this we have to prove

Because the part taken of the blue line is equal to the part cut off from the red line, according to the Construction, and the yellow line is part of each of the two triangles the one of which is formed by the part taken of the blue line, the yellow line, and the dotted blue line, and the other by the part cut off from the red line, the yellow line, and the dotted red line, therefore the two sides, viz the part of the blue line and the yellow line, are equal to the two sides, viz. the part of the red line and the yellow line each to each, and the base,

viz. the dotted blue line, is equal to the base, viz. the dotted red line, because we made it so; consequently, according to Proposition VIII., the angle formed by the meeting of the blue line and the yellow line is equal to the angle formed by the meeting of the red line and the yellow line. And these two angles together make up the given rectilineal angle (i.e. an angle contained by straight lines), therefore the given angle, viz. the angle formed by the meeting of the blue line and the red line, is bisected (i.e. divided into halves) by the yellow line: which is what we had to do.

PROPOSITION X.—PROBLEM.

To bisect a given finite straight line, i.e. to divide it into two equal parts.

First, we have to draw a straight line, which we have to

divide into two equal parts.

Second, as shown in Proposition I., we make upon it an equilateral triangle.

Third, as shown in Proposition IX., we bisect (i.e. divide into halves) the angle formed by the meeting of the blue line and the red line, by the straight yellow line.

The black line shall be cut into two equal parts at the

point where the yellow line meets it This we have to prove

Because the blue line is equal to the red line (according to the Construction), and the yellow line is part of each of the two triangles the one of which is formed by the blue line, the yellow line, and part of the black line, and the other by the red line, the yellow line, and the remainder of the black line, the two sides, viz the blue line and the yellow line, are equal to the two sides, viz the red line and the yellow line, each to each, and the angle formed by the meeting of the blue line and the yellow line is equal to the angle formed by the meeting of the red line and the yellow line (because we made it so), therefore, according to Proposition IV, the base, viz the one part of the black line, is equal to the base, viz the other part of the black line, and therefore the straight black line is divided into two equal parts at the point where the yellow line meets it, which is what was to be done

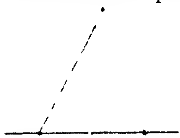
Proposition XI —Problem

To draw a straight line at right angles to a given straight line from a given point in the same

First, we draw the straight line, and mark the point in it from which we are to draw a straight line at right angles to it

Second, we take any point in the black line on either side of the point already marked, and from the black line on the other side of the point already marked we cut off, according to Proposition III, a part equal to the part between the given point and the other point

Third, upon the whole line between the two points thus taken we make an equilateral triangle, as shown in Proposition I



Fourth, we join the point where the red side and the blue side of the triangle meet, with the point from which the line is to be drawn at right angles

The yellow line shall be at right angles to the black line and we have to prove it so

Because the part taken of the black line on one side of the given point is equal to the part found in the same line on the other side of the point, and the yellow line is part of each of the two triangles the one of which is formed by the part taken of the black line, the yellow line, and the red line, and the other by the corresponding part taken of the black line, the yellow line, and the blue line, the two sides, viz the one part taken of the black line, and the yellow line, are equal to the two sides, viz the other part taken of the black line, and the yellow line, each to each, and the base, viz the red line, is equal to the base, viz the blue line, by the Construction, therefore, according to Proposition VIII, the angle formed by the meeting of the one part taken of the black line with the yellow line is equal to the angle formed by the meeting of the other part taken of the black line with the yellow line, and they are adjacent (adjoining) angles

But we know by Definition X that when the adjacent angles which one straight line makes with another straight line are equal to one another, each of them is called a right angle Therefore each of the two angles formed by the meeting of the yellow line with the black line is a right angle And the yellow line making them right angles is drawn

from the point marked in the black line which is what was to be done

Corollary (i e another fact proved) —By help of this problem it may be shown that two straight lines cannot have a part the same in both

First, if it be possible let us draw two straight lines having a part the same in both

Second, from the point

where the part of the line which is the same in both lines ends, draw the blue line at right angles to that part, as shown in Proposition XI

Because the line composed of the black line and the dotted black line is a straight line, the angle formed by the meeting of the blue line with the dotted black line is equal to the angle formed by the meeting of the blue line with the black line, according to Definition X, which says that when a straight line standing on another straight line makes the adjacent (adjoining) angles equal, each of these angles is called a right angle In the same manner, because the line composed of the black line and the dotted red line is a straight line, the angle formed by the meeting of the dotted red line with the blue line is equal to the ang'e formed by the meeting of the black line with the blue line It follows, because, according to Axiom I, things which are equal to the same thing are equal to one another, that the angle formed by the meeting of the dotted red line with the blue line is equal to the angle formed by the meeting of the dotted black line with the blue line, i e the less is equal to the greater, which is plainly absurd, and therefore two straight lines cannot have a part the same in both which is what we had to prove

Proposition XII.—Problem

To draw a straight line perpendicular to a given straight line of an unlimited length, from a given point without it.

First, we draw the line upon which we are to draw a perpendicular straight line (i.e. a line at right angles to it), and mark the point beyond the line from which the perpendicular is to be drawn.

Second, we take any point the other side of the straight line.

Third, from the first point as centre, with the other point as distance, we describe a circle as allowed by Postulate III., which says that 'a circle may be described from any centre at any distance from that centre, meeting the line in two points.

Fourth, we bisect (i.e. divide into halves) the line between the points where the circle meets it, as shown in Proposition X., and join the blue point with the point where the line is bisected. Then the blue line shall be perpendicular to the black line and we have to prove it.

Fifth, join the blue point with each of the points where the circle meets the black line

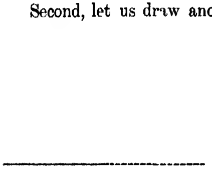
Because the one black line from the blue line to the yellow circle is equal to the other black line from the blue line to the yellow circle (for we made it so), and the blue line is part of the two triangles the one of which is formed by the one black line, the blue line, and the red line, and the other by the other black line, the blue line, and the yellow line, therefore the two sides, viz the one black line and the blue line, are equal to the two sides, viz the other black line and the blue line, each to each, and the base, viz the red line, is equal to the base, viz the yellow line, according to Definition XV, which says that all straight lines drawn from the centre of a circle to its circumference are equal to one another, therefore, as shown in Proposition VIII, the angle formed by the meeting of the one black line with the blue line is equal to the angle formed by the meeting of the other black line with the blue line, and they are adjacent (adjoining) angles But we know by Definition X that when a straight line standing on another straight line makes the adjacent angles equal to one another, each of them is a right angle, and the straight line which stands upon the other is called a perpendicular to it Therefore the blue line is perpendicular to the black line And it is drawn from the point from which it was to be drawn so that we have done what was required

Proposition XIII —Theorem

The angles which one straight line makes with another upon one side of it are either two right angles or are together equal to two right angles

First, let us draw one straight line

Second, let us draw another straight line making with the straight line we first drew and upon the same side of it two angles These angles shall be either two right angles or shall be together equal to two right angles



For if the angle made by the meeting of the black part of the line with the blue line be equal to the angle made by the meeting of the dotted part of the line with the blue line, we know each of them is a right angle, according to Definition X, which says that when a straight line standing on another straight line makes the adjacent (adjoining) angles equal, each of these angles is called a right angle

But if the two angles be not equal, from the point where the blue line meets the black line draw, as shown in Proposition XI, the red line at right angles to the black line therefore, according to Definition X just quoted, the angle made by the meeting of the red line and the black part of the black line, and

the angle made by the meeting of the dotted part of the black line and the red line, are two right angles And because the angle made by the meeting of the dotted part of the black

line and the red line is equal to the two angles made by the meeting of the dotted part of the black line with the blue line, and the meeting of the blue line with the red line together, add to each of these equals the angle made by the meeting of the red line with the black part of the black line Therefore, according to Axiom II, which says that if equals be added to equals the wholes are equal, the angles formed by the meeting of the dotted part of the black line with the red line, and the meeting of the red line with the black part of the black line, are together equal to the three angles formed by the meeting of the dotted part of the black line with the blue line, the meeting of the blue line with the red line, and the meeting of the red line with the black part of the black line Again, because the angle made by the meeting of the black part of the black line with the blue line is equal to the two angles made by the meeting of the black part of the black line with the red line, and the meeting of the red line with the blue line together, to each of these equals add the angle made by the meeting of the blue line with the dotted part of the black line Then, according to Axiom II, which says that if equals be added to equals the wholes are equal, the two angles formed by the meeting of the black part of the black line with the blue line, and the meeting of the blue line with the dotted part of the black line, are together equal to the three angles made by the meeting of the black part of the black line with the red line, the meeting of the red line with the blue line, and the meeting of the blue line with the dotted part of the black line But the two angles formed by the meeting of the dotted part of the black line with the red line, and the meeting of the red line with the black part of the black line, have been proved to be together equal to the same three angles And we know by Axiom I that things which are equal to the same thing are equal to one another, therefore the angles formed by the meeting of the dotted part of the black line

with the red line, and the meeting of the red line with the black part of the black line, are equal to the angles formed by the meeting of the black part of the black line with the blue line, and the meeting of the blue line with the dotted part of the black line But the angles formed by the meeting of the dotted part of the black line with the red line, and the meeting of the red line with the black part of the black line, are two right angles, because we made them so therefore the angles formed by the meeting of the black part of the black line with the blue line, and the meeting of the blue line with the dotted part of the black line, are together equal to two right angles, according to Axiom I, which says that things which are equal to the same thing are equal to one another We have thus proved what was required

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PROPOSITION XIV —THEOREM

If, at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent (adjoining) angles together equal to two right angles, these two straight lines shall be in one and the same straight line

First, let us draw a straight line

Second, let us draw two straight lines at the end of the

straight line, upon opposite sides of it, making the adjacent (adjoining) angles at the point where they meet the black line equal together to two right angles.

The red line shall be in the same straight line with the blue line, and this we shall have to prove

For if the red line be not in the same straight line with the blue line, let us draw a yellow line in the same straight line with the blue line

Then, because the straight black line makes with the straight blue and yellow line, upon one side of it, the two angles, viz the angle at the point where it meets the blue part of the line and the angle at the point where it meets the yellow part of the line, these angles are together equal to two right angles, according to Proposition XIII But the angle at the point where the black line meets the blue part of the line, and the angle at the point where the black line meets the red line, are likewise together equal to two right angles, according to the Hypothesis It therefore follows, according to Axiom II, which says that things which are equal to the same thing are equal to one another, that the angle at the point where the black line meets the blue part of the line, and the angle at the point where the black line meets the yellow part of the line, are together equal to the angle at the point where the black line meets the blue line and the angle at the point where the black line meets the red line Now let us take away from each of these equals the angle at the point where the black line meets the blue line, which is part of them both, then, according to Axiom III, which says that if equals be taken from equals the remainders are equal, the remaining angle at the point where the black line meets the yellow part of the line is equal to the remaining angle at the point where the black line meets the red line, the less to the greater, which is impossible Therefore the yellow line is not in the same straight line with the blue line In the same way it may

be shown that no other line but the red line can be in the same straight line with the blue line Therefore the red line is in the same straight line with the blue line and this is what we had to prove

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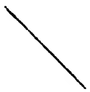
Proposition XV —Theorem

If two straight lines cut one another, the vertical, or opposite, angles shall be equal

First, we draw two straight lines cutting one another

Then the angle at the point where the red part of the one line meets the yellow part of the other line shall be equal to the angle at the point where the blue part of the one line meets the black part of the other line, and the angle at the point where the yellow part of one line meets the blue part of the other line shall be equal to the angle at the point where the black part of the one line meets the red part of the other line This we have to prove



Because the straight red line makes with the straight yellow and black line the angle at the point where it meets the yellow part of the line and the angle at the point where it meets the black part of the line, these angles are together equal to two right angles, as shown in Proposition XIII Again, because the straight black line makes with the straight red and blue line the angle at the point where it meets the red part of the line, and the angle at the point where it meets the blue part of the line, these two angles are together equal to two right angles, as shown in Proposition XIII

And the angle at the point where the straight red line meets the yellow part of the line, and the angle at the point where it meets the black part of the line, have been shown to be equal to two right angles, therefore, according to Axiom I, which says that things which are equal to the same thing are equal to one another, the angle at the point where the red line meets the yellow part of the line, and the angle at the point where it meets the black part of the line, are equal to the angle at the point where the black line meets the red part of the line, and the angle at the point where it meets the blue part of the line Take away the angle at the point where the red line meets the black part of the line, which is part of each of these equals, and it follows, according to Axiom III, which says that if equals be taken from equals the remainders are equal, that the remaining angle at the point where the red line meets the yellow part of the line is equal to the remaining angle at the point where the black line meets the blue part of the line In the same manner it can be shown that the angle at the point where the yellow line meets the blue part of the line is equal to the angle at the point where the black line meets the red part of the line And each pair of these equal angles are vertical, i e. opposite, angles so that we have proved what we had to prove

Corollary 1 (i e another fact proved)—From this it is plain that if two straight lines cut one another, the angles which they make at the point where they cut, are together equal to four right angles

Corollary 2 (i e another fact proved)—And therefore that all the angles made by any number of lines meeting in one point are together equal to four right angles

Proposition XVI.—Theorem

If one side of a triangle be produced (lengthened), the exterior (outside) angle is greater than either of the interior (inside) opposite angles.

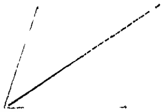
First, let us draw a triangle.

Second, let us produce (lengthen) the blue side of the triangle.

Then the exterior (outside) angle at the point where the yellow line meets the dotted blue line shall be greater than either of the interior (inside) opposite angles at the point where the red line meets the yellow line and at the point where it meets the blue line.

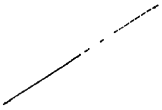
Third, bisect (i. e. equally divide) the yellow line, as shown in Proposition X., and join the point of division with the point where the red side and the blue side meet.

Fourth, produce (lengthen) the black line, and make the produced part equal to the black line, as shown in Proposition III.



Fifth, join the end of the produced black line with the point where the yellow line and the blue line meet.

Because the thick part of the yellow line is equal to the thin part of the yellow line (for we made it so), and the black line is equal to the dotted black line (for we made it so), the thick part of the yellow line and the black line are equal to the thin part of the yellow line and the dotted black line, each to each and we know by



Proposition XV that the angle at the point where the thick part of the yellow line meets the black line is equal to the angle at the point where the thin part of the yellow line meets the dotted black line because they are opposite vertical angles, therefore, as shown in Proposition IV, the base, viz the red line, is equal to the base, viz the dotted red line, and the triangle formed by the thick part of the yellow line, the black line, and the red line, is equal to the triangle formed by the thin part of the yellow line, the dotted black line, and the dotted red line, and the remaining angles are equal to the remaining angles, each to each, to which the equal sides are opposite Therefore the angle at the point where the red line meets the thick part of the yellow line is equal to the angle at the point where the thin part of the yellow line meets the dotted red line But the angle at the point where the thin part of the yellow line meets the dotted blue line is greater than the angle at the point where the thin part of the yellow line meets the dotted red line, according to Axiom IX, which says that the whole is greater than its part Therefore the angle at the point where the yellow line meets the dotted blue line is greater than the angle at the point where the red line meets the thick part of the yellow line

In the same manner, if the blue side be bisected (i e equally divided) and the yellow side produced, it may be shown that the angle at the point where the blue line

meets the dotted yellow line, and consequently, according to Proposition XV, the angle at the point where the thin part of the yellow line meets the dotted blue line (the opposite vertical angle), is greater than the angle at the point where the red line meets the blue line We have therefore proved what we had to prove

Proposition XVII —Theorem

Any two angles of a triangle are together less than two right angles

First, we draw a triangle We have to prove that any two of its angles are together less than two right angles

Second, we produce (lengthen) the black side to any length

Then because the angle at the point where the dotted black line meets the red line is the exterior (outside) angle of the triangle formed by the blue, the black, and the red lines, it follows, as shown in Proposition XVI, that it is greater than the interior (inside) and opposite angle at the point where the blue line meets the black line To each of these angles add the angle at the point where the red line

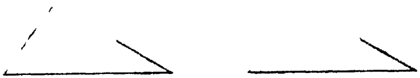
meets the black line It is then true, according to Axiom IV, which says that if equals be added to unequals the wholes are unequal, that the angle at the point where the red line meets the dotted black line, and the angle at the point where it meets the black line, are together greater than the angle at the point where the blue line meets the black line, and the angle at the point where the red line meets the black line But we know by what we proved in Proposition XIII that the angle at the point where the red line meets the black line, and the angle at the point where it meets the dotted black line, are together equal to two right angles therefore the angle at the point where the blue line meets the black line, and the angle at the point where the red line meets the black line, are together less than two right angles And these are two interior angles of the triangle In the same way, by producing the blue side and the red side of the triangle, it may be shown that the angle at the point where the blue line meets the red line, and the angle at the point where it meets the black line, are together less than two right angles, and that the angle at the point where the red line meets the black line, and the angle at the point where it meets the blue line, are together less than two right angles. We have therefore proved what we had to prove

Proposition XVIII.—Theorem

The greater side of every triangle is opposite to the greater angle

First, let us draw a triangle, the red side of which is greater than the blue side We have to prove that the angle at the point where the blue line meets the black line is greater than the angle at the point where the red line meets the black line.

Because the red line is greater than the blue line (for we made it so), cut off from it, as shown in Proposition III, a part equal to the blue line, and join the end of the part cut off with the point where the blue line meets the black line



Then because the angle at the point where the thin part of the red line meets the yellow line is the exterior (outside) angle of the triangle formed by the yellow line, the thick part of the red line, and the black line, it is greater, as shown in Proposition XVI, than the interior (inside) and opposite angle at the point where the red line meets the black line But the angle at the point where the thin part of the red line meets the yellow line is equal to the angle at the point where the blue line meets the yellow line, because the thin part of the red line is equal to the blue line (for we made it so) therefore the angle at the point where the blue line meets the yellow line is also greater than the angle at the point where the red line meets the black line Much more then is the angle at the point where the blue line meets the black line greater than the angle at the point where the red line meets the black line We have therefore proved what we had to prove

Proposition XIX —Theorem

The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it

First, we draw a triangle, making the angle at the point where the blue side of the triangle meets the black side

greater than the angle at the point where the red side meets the black side We have to prove that the red side is greater than the blue side

If it be not greater, the red side must be either equal to or less than the blue side It is not equal, because then the angle at the point where the red side meets the black side would be equal to the angle at the point where the blue side meets the

black side, as shown in Proposition V, and it is not, because we made it less Therefore the red line is not equal to the blue line Neither is the red line less than the blue line, because then, as shown in Proposition XVIII, the angle at the point where the blue line meets the black line would be less than the angle at the point where the red line meets the black line, but it is not, because we made it greater Therefore the red line is not less than the blue line And it has been shown that it is not equal to it, hence the red line must be greater than the blue line which is what we had to prove

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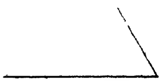
Proposition XX —Theorem

Any two sides of a triangle are together greater than the third side

First, we draw a triangle We have to prove that any two sides of it are together greater than the third side, viz the blue side and the red side greater than the black side, the blue side and the black side greater than the



red side, and the black side and the red side greater than the blue side

Second, produce the blue line, and make the dotted blue line equal to the red line, as shown in Proposition III

Third, join the end of the dotted blue line with the point where the red line meets the black line

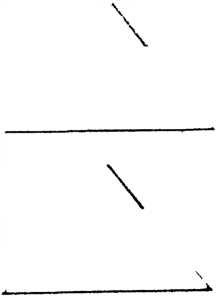
Because the dotted blue line is equal to the red line (for we made it so), the angle at the point where the dotted blue line meets the dotted red line is equal to the angle at the point where the red line meets the dotted red line, as shown in Proposition V But the angle at the point where the black line meets the dotted red line is greater than the angle at the point where the red line meets the dotted red line, according to Axiom IX, which says that the whole is greater than its part, therefore the angle at the point where the black line meets the dotted red line is greater than the angle at the point where the dotted blue line meets the dotted red line And because the angle at the point where the black side meets the dotted red side of the triangle formed by the dotted red line, the black line, and the blue and dotted blue line, is greater than the angle at the point where the blue and dotted blue line meets the dotted red line, and that, as shown in Proposition XIX, the greater angle is subtended by (i e has opposite to it) the greater side, therefore the blue and dotted blue side is greater than the black side But the blue and dotted blue line is equal to the blue line and the red line together, because the dotted blue line being equal to the red line (for we made it so), if we add the blue line to each, it follows from Axiom II,

which says that if equals be added to equals the wholes are equal, that the whole blue and dotted blue line is equal to the blue line and the red line together Therefore the blue side and the red side are together greater than the black side In the same way we can prove that the blue side and the black side are together greater than the red side, and the black side and the red side together greater than the blue side We have therefore proved what we had to prove

Proposition XXI —Theorem

If from the ends of the side of a triangle there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle

First, we draw a triangle, and mark a point within it

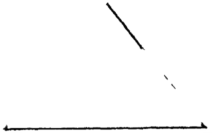


Second, we draw two straight lines, one from each end of the black line, to the point marked within the triangle We have to prove that the dotted blue line and the dotted red line are less than the blue line and the red line, but that the angle at the point where the dotted blue line and the dotted red line meet is greater than the angle at the point where the blue line and the red line meet

Third, we produce the dotted blue line to meet the red line.

Then because two sides of a triangle are greater than the

third side, as shown in Proposition XX., the blue side and the thick red side of the triangle formed by the blue line, the thick red line, and the dotted blue and yellow line, are greater than the dotted blue and yellow line. To each of these add the thin red line. Then, according to Axiom IV., which says that if equals be added to unequals the wholes are unequal, the two sides, viz. the blue side and the side composed of the thick red line and the thin red line, are greater than the two sides, viz. the side composed of the dotted blue line and the yellow line, and the thin red line. Again, because, according to Proposition XX., the thin red side and the yellow side of the triangle formed by the thin red line, the yellow line, and the dotted red line, are greater than the dotted red line, add the dotted blue line to each of these. Then, according to Axiom IV., which says that if equals be added to unequals the wholes are unequal, the two sides, viz. the thin red side and the side composed of the yellow line and the dotted blue line, are greater than the dotted red line and the dotted blue line. But we have shown that the two sides, viz. the blue side and the side composed of the thick red line and the thin red line, are greater than the two sides, viz. the side composed of the dotted blue line and the yellow line, and the thin red line: much more then are the two sides, viz. the blue side and the side composed of the thick red line and the thin red line, greater than the two sides, viz. the dotted blue line and the dotted red line: which is the first part of what we had to prove.



Again, because, as shown in Proposition XVI., the exterior (outside) angle of a triangle is greater than the interior (inside) and opposite angle, the exterior angle at the point where the dotted blue line and the dotted red line

meet, of the triangle formed by the dotted red line, the yellow line, and the thin red line, is greater than the angle at the point where the thin red line and the yellow line meet For the same reason, the exterior angle at the point where the thin red line and the line composed of the dotted blue line and the yellow line meet, of the triangle formed by the blue line, the dotted blue and the yellow line, and the thick red line, is greater than the angle at the point where the blue line meets the red line And we have shown that the angle at the point where the dotted blue line meets the dotted red line is greater than the angle at the point where the thin red line and the line composed of the dotted blue line and the yellow line meet, therefore much more is the angle at the point where the dotted blue line and the dotted red line meet greater than the angle at the point where the blue line and the red line meet which is the other part of what we had to prove

PROPOSITION XXII —PROBLEM

To make a triangle of which the sides shall be equal to three given straight lines, but any two of these lines must be greater than the third (according to Proposition XX)

First, we draw the three straight lines, of which any two whatever are greater than the third, viz the red line and the blue line greater than the black line, the red line and the black line greater than the blue line, and the blue line and the black line greater than the red line

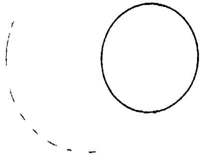
What we have to do is to make a triangle of which the sides shall be equal to the red line, the blue line, and the black line, each to each

Second, we draw another straight line, beginning from

some fixed point, but of an unlimited length in the other direction, and, as shown in Proposition III, make the thick yellow part equal to the red line, the thin yellow part equal to the blue line, and the dotted yellow part equal to the black line

Third, from the point where the thick yellow part of the line meets the thin yellow part of the line as centre, at a distance equal to the thick yellow part, we make the red circle as allowed by Postulate III, which says that a circle may be made from any centre, at any distance from that centre

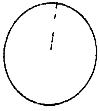
Fourth, from the point where the thin yellow part of the line meets the dotted yellow part of the line as centre, at a distance equal to the thin dotted line, we make the black circle as allowed by Postulate III, just quoted



Fifth, we join either of the points where the red circle cuts the black circle, with the two ends of the thin part of the yellow line Then the triangle formed by the dotted red line, the dotted black line, and the thin part of the yellow line, shall have its sides equal to the three straight lines, the red line, the black line, and the blue line

Because the point where the thick part of the yellow line meets the dotted red line is the centre of the red circle, the thick part of the yellow line is equal to the dotted red line, according to Definition XV, which says that all

straight lines drawn from the centre of a circle to the circumference are equal to one another But the thick part of the yellow line is equal to the straight red line (for we made it so), therefore, according to Axiom I, which says that things which are equal to the same thing are equal to one another, the dotted red line is equal to the straight red line



Again, because the point where the dotted part of the yellow line meets the dotted black line is the centre of the black circle, the dotted part of the yellow line is equal to the dotted black line, according to Definition XV just quoted But the dotted part of the yellow line is equal to the straight black line (for we made it so), therefore, according to Axiom I just quoted, the dotted black line is equal to the straight black line And the thin part of the yellow line we made equal to the straight blue line, therefore the three straight lines, viz the dotted red line, the thin part of the yellow line, and the dotted black line, are equal to the three, viz the red line, the blue line, and the black line, and consequently the triangle formed by the dotted red line, the thin part of the yellow line, and the dotted black line, has its three sides equal to the three given straight lines which is what we had to do

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Proposition XXIII —Problem

At a given point in a given straight line to make a rectilineal angle (i e an angle contained by straight lines) equal a given rectilineal angle.

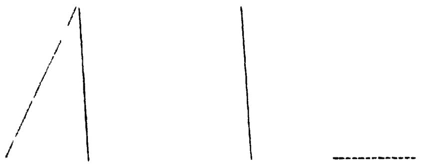
First, we draw the straight line and take the upper end of it as the point at which the angle is to be made

Second, we draw the angle which is the given rectilineal angle (i e an angle contained by straight lines)

What we have to do is to make at the upper end of the red line an angle equal to the angle at the point where the blue line meets the black line

Third, in the blue line we take any point, and in the black line take any point and join those points

Fourth, as shown in Proposition XXII, we make the triangle formed by the red line, the dotted blue line, and the dotted black line, the sides of which shall be equal to the three straight lines, the blue line, the black line, and the yellow line, viz the red side to the blue line, the dotted black side to the yellow line, and the dotted blue side



to the black line The angle at the point where the red line meets the dotted blue line shall be equal to the angle at the point where the blue line meets the black line

Because the blue line and the black line are equal to the red line and the dotted blue line, each to each, and the base, viz the yellow line, to the base, viz the dotted black line (for we made them so), it follows, as shown in Proposition VIII,

that the angle at the point where the blue line and the black line meet is equal to the angle at the point where the red line and the dotted blue line meet We have therefore made an angle at the upper end of the red line equal to the given angle, as required

Proposition XXIV —Theorem

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them of the other, the base of the triangle which has the greater angle shall be greater than the base of the other

First, we draw two triangles, having the red side and the blue side of the one equal to the dotted red side and the

dotted blue side of the other, each to each, viz the red side to the dotted red side, and the blue side to the dotted blue side, but the angle at the point where the red side and the blue side meet greater than the angle at the point where the dotted red side and the dotted blue side meet we have to prove that the base, viz the black side, is greater than the base, viz the dotted black side

Second, of the two sides, viz the dotted red side and the dotted blue side, let the dotted red side be the one which is

not greater than the other, and at the point where these two sides meet, make, as shown in Proposition XXIII, the angle

formed by the dotted yellow line and the dotted red side equal to the angle at the point where the blue side meets the red side, and, as shown in Proposition III, make the dotted yellow line equal to the blue side, or the dotted blue side.

Third, we join the lower end of the dotted yellow line with the point where the dotted red side meets the dotted black side, and with the point where the dotted blue side meets the dotted black side.

Because the red side is equal to the dotted red side (by the Hypothesis), and the blue side equal to the dotted yellow line (for we made it so), the two sides, viz. the red side and the blue side, are equal to the two sides, viz. the dotted red side and the dotted yellow line, each to each, and the angle at the point where the red side and the blue side meet is equal to the angle at the point where the dotted red side and the dotted yellow line meet (for we made it so), therefore, as shown in Proposition IV, the base, viz. the thick black line, is equal to the base, viz. the thin black line.

And because the dotted yellow line is equal to the dotted blue side, the angle at the point where the dotted blue side meets the yellow line is equal to the angle at the point where the dotted yellow line meets the yellow line, as shown in Proposition V. But the angle at the point where the dotted yellow line meets the yellow line is greater than the angle at the point where the thin black line meets the yellow line, according to Axiom IX, which says that the whole is greater than its part: therefore the angle at the point where the dotted blue line meets the yellow line is

greater than the angle at the point where the thin black line meets the yellow line therefore much more is the angle at the point where the dotted black side meets the yellow side greater than the angle at the point where the thin black line meets the yellow line And because, in the triangle formed by the dotted black line, the yellow line, and the thin black line, the angle at the point where the dotted black line meets the yellow line is greater than the angle at the point where the thin black line meets the yellow line, and that, as shown in Proposition XIX, the greater angle is subtended by (i e has opposite to it) the greater side, therefore the thin black side is greater than the dotted black side But the thin black side was proved to be equal to the thick black side, therefore the thick black side is greater than the dotted black side which is what we had to prove

Proposition XXV—Theorem

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other, the angle contained by the sides of that which has the greater base shall be greater than the angle contained by the sides equal to them, of the other

First, let us make two triangles, of which the red side and the blue side of the one are equal to the dotted red side and

the dotted blue side of the other, each to each, viz the red side to the dotted red side, and the blue side to the dotted blue

side, but the base, viz the black side, greater than the base, viz the dotted black side Then we have to prove that the angle at the point where the red side and the blue side meet is greater than the angle at the point where the dotted red side and the dotted blue side meet

If it be not greater, it must either be equal to or less than it But the angle at the point where the red side meets the blue side is not equal to the angle at the point where the dotted red side meets the dotted blue side, because if it were, then, according to Proposition IV, the base, viz the black side, would be equal to the base, viz the dotted black side, which it is not, according to the Hypothesis Neither is it less, because if it were, then, according to Proposition XXIV, the base, viz the black side, would be less than the base, viz the dotted black side, which it is not, according to the Hypothesis Therefore, as the angle at the point where the red side meets the blue side is neither equal to the angle at the point where the dotted red side meets the dotted blue side, nor less than it, it is greater than it which is what we had to prove

Proposition XXVI —Theorem

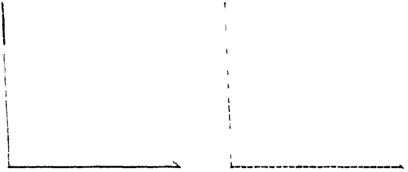
If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side viz either the sides adjacent to (adjoining) the equal angles, or the sides opposite to equal angles in each, then shall the other sides be equal, each to each, and also the third angle of the one to the third angle of the other

First, let us make two triangles which have the angle at the point where the red side meets the black side, and the angle at the point where the blue side meets the black side, equal to the angle at the point where the dotted red side meets the dotted black side, and the angle at the point where the dotted blue side meets the dotted black side, each to

each, viz the angle at the point where the red side meets the black side, equal to the angle at the point where the dotted

red side meets the dotted black side, and the angle at the point where the blue side meets the black side equal to the angle at the point where the dotted blue side meets the dotted black side Also one side equal to one side and first let those sides be equal which are adjacent to (adjoining) the angles that are equal in the two triangles, viz the black side to the dotted black side Then the other sides shall be equal, each to each, viz the red side to the dotted red side, and the blue side to the dotted blue side, and the third angle at the point where the red side and the blue side meet, equal to the third angle at the point where the dotted red side meets the dotted blue side

For if the red side be not equal to the dotted red side, one of them must be greater than the other Let the red



side be the greater of the two, and, as shown in Proposition III., cut off a part of it equal to the dotted red side, and

join the end of the part thus cut off with the point where the blue side and the black side meet Then because the thin part of the red side is equal to the dotted red side, and the black side to the dotted black side (by the Hypothesis), the two sides, viz the thin part of the red side and the black side, are equal to the two sides, viz the dotted red side and the dotted black side, each to each And the angle at the point where the thin part of the red side meets the black side is equal to the angle at the point where the dotted red side meets the dotted black side (by the Hypothesis), therefore, as shown in Proposition IV, the base, viz the yellow side, is equal to the base, viz the dotted blue side, and the triangle formed by the thin part of the red side, the black side, and the yellow side, to the triangle formed by the dotted red side, the dotted black side, and the dotted blue side, and the other angles to the other angles, each to each, to which the equal sides are opposite Therefore the angle at the point where the yellow side meets the black side is equal to the angle at the point where the dotted blue side meets the dotted black side But the angle at the point where the dotted blue side meets the dotted black side is, by the Hypothesis, equal to the angle at the point where the blue side meets the black side therefore, according to Axiom I, which says that things which are equal to the same thing are equal to one another, the angle at the point where the black side meets the yellow side is equal to the angle at the point where the black side meets the blue side, the less to the greater, which is impossible Therefore the whole red side is not unequal to the dotted red side, i e it is equal to it, and the black side is equal to the dotted black side, by the Hypothesis, therefore the two, viz the whole red side and the black side, are equal to the two, viz the dotted red side and the dotted black side, each to each, and, by the Hypothesis, the angle at the point where the red side meets the black side is equal to the angle at the point where the

dotted red side meets the dotted black side, therefore, as shown in Proposition IV, the base, viz. the blue side, is equal to the base, viz. the dotted blue side, and the third angle, at the point where the red side meets the blue side, to the third angle, at the point where the dotted red side meets the dotted blue side: which is what we had to prove.

Next, let the sides which are opposite to equal angles in each triangle be equal to one another, viz. the red side to the dotted red side: likewise, in this case, the other sides shall be equal, viz. the blue side to the dotted blue side, and the black side to the dotted black side, and also the third angle, at the point where the red side meets the blue side, to the third angle, at the point where the dotted red side meets the dotted blue side.

For if the black side be not equal to the dotted black side, let the black side be the greater of them, and from it,

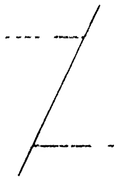
as shown in Proposition III, cut off a part equal to the dotted black side, and join the end of the part thus cut off with the point where the red side meets the blue side. Then because the thin part of the black side is equal to the dotted black side, and the red side to the dotted red side (by the Hypothesis), the two, viz. the red side and the thin part of the black side, are equal to the two, viz. the dotted red side and the dotted black side, each to each, and by the Hypothesis the angle at the point where the red side meets the black side is equal to the angle at the point where the

dotted red side meets the dotted black side, therefore, as shown in Proposition IV, the base, viz the yellow side, is equal to the base, viz the dotted blue side, and the triangle formed by the red side, the thin part of the black side, and the yellow side, to the triangle formed by the dotted red side, the dotted black side, and the dotted blue side, and the other angles to the other angles, each to each, to which the equal sides are opposite therefore the angle at the point where the yellow side meets the black side is equal to the angle at the point where the dotted blue side meets the dotted black side But, by the Hypothesis, the angle at the point where the dotted blue side meets the dotted black side is equal to the angle at the point where the blue side meets the black side, therefore, according to Axiom I, which says that things which are equal to the same thing are equal to one another, the angle at the point where the yellow side meets the black side is equal to the angle at the point where the blue side meets the black side, i e the exterior (outside) angle of the triangle formed by the yellow side, the thick part of the black side, and the blue side, is equal to its interior (inside) angle, which is impossible, as shown in Proposition XVI Consequently the whole black side is not unequal to the dotted black side, i e it is equal to it And the red side is by the Hypothesis equal to the dotted red side, therefore the two, viz the red side and the whole black side, are equal to the two, viz the dotted red side and the dotted black side, each to each And the angle at the point where the red side meets the black side is by the Hypothesis equal to the angle at the point where the dotted red side meets the dotted black side, therefore, as shown in Proposition IV, the base, viz the blue side, is equal to the base, viz the dotted blue side, and the third angle, at the point where the red side meets the blue side, to the third angle, at the point where the dotted red side meets the dotted blue side. which is what we had to prove.

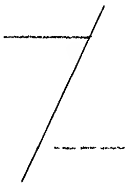
Proposition XXVII.—Theorem

If a straight line falling upon two other straight lines makes the alternate angles equal to one another, these two straight lines shall be parallel.

First, let us draw the straight black line, which falls upon the two straight lines, viz. the red line and the blue line, and makes the alternate angles, viz. the angle at the point where the thick part of the red line meets the black line, and the angle at the point where the black line meets the thick part of the blue line, equal to one another: then we have to prove that the red line is parallel to the blue line.



If the two lines be not parallel, they will meet if produced either on the right hand or on the left hand. Let us produce them and suppose them to meet on the right hand.



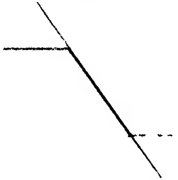
Then the figure formed by the thin part of the red line, the dotted red line, the dotted blue line, and the thick part of the blue line, is a triangle, and, as shown in Proposition XVI., its exterior angle, at the point where the thick part of the red line meets the black line, is greater than the interior and opposite angle, at the point where the thick part of the blue line meets the black line. But according to the Hypothesis it is equal to it; therefore it is impossible for it to be greater than it, and consequently

the red line and the blue line being produced do not meet on the right hand In like manner, it can be shown they do not meet on the left hand But, according to Definition XXXV , those straight lines which meet neither way though produced ever so far, are parallel to one another therefore the red line is parallel to the blue line which is what we had to prove

Proposition XXVIII —Theorem

If a straight line falling upon two other straight lines makes the exterior angle equal to the interior and opposite angle upon the same side of the line, or makes the interior angles upon the same side together equal to two right angles, the two straight lines shall be parallel to one another

First, let us draw the straight black line which falls upon the two straight lines, viz the red line and the blue line,



and makes the exterior angle at the point where the thin part of the black line meets the thin part of the red line, equal to the interior and opposite angle at the point where the thick part of the black line meets the thick part of the blue line, upon the same side or makes the interior angles on the same side, viz the angle at the point where the thin part of the red line meets the thick part of the black line, and the angle at the point where the thick part of the blue line meets the thick part of the black line, together equal to two right angles. We have to prove that the red line is parallel to the blue line.

Because the angle at the point where the thin part of the black line meets the thin part of the red line is equal to the angle at the point where the thick part of the black line meets the thick part of the blue line (according to the Hypothesis), and the angle at the point where the thin part of the black line meets the thin part of the red line is also equal, as shown in Proposition XV, to the angle at the point where the thick part of the red line meets the thick part of the black line, therefore the angle at the point where the thick part of the red line meets the thick part of the black line is equal to the angle at the point where the thick part of the black line meets the thick part of the blue line, according to Axiom I, which says that things which are equal to the same thing are equal to one another And these two angles are alternate angles therefore, as shown in Proposition XXVII, the red line is parallel to the blue line

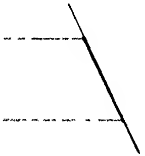
Again, because the angle at the point where the thin part of the red line meets the thick part of the black line, and the angle at the point where the thick part of the blue line meets the thick part of the black line, are together equal to two right angles (according to the Hypothesis), and that the angle at the point where the thick part of the red line meets the thick part of the black line, and the angle at the point where the thin part of the red line meets the thick part of the black line, are also together equal to two right angles, therefore the angle at the point where the thick part of the red line meets the thick part of the black line, and the angle at the point where the thin part of the red line meets the thick part of the black line are equal to the angle at the point where the thin part of the red line meets the thick part of the black line, and the angle at the point where the thick part of the black line meets the thick part of the blue line, according to Axiom I, which says that things which are equal to the same thing are equal to one

another Let us now take away the angle at the point where the thin part of the red line meets the thick part of the black line, which is part of each of these equals, and it follows, according to Axiom III, which says that if equals be taken from equals, the remainders are equal, that the remaining angle at the point where the thick part of the red line meets the thick part of the black line is equal to the remaining angle at the point where the thick part of the black line meets the thick part of the blue line And these are alternate angles, therefore, as shown in Proposition XXVII, the red line is parallel to the blue line which is what we had to prove

Proposition XXIX —Theorem

If a straight line fall upon two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite angle upon the same side, and likewise the two interior angles upon the same side together equal to two right angles

First, we draw the straight black line falling upon the parallel straight lines, the blue line, and the red line We have to prove that the alternate angles, viz the angle at the point where the thick part of the red line meets the thick part of the black line, and the angle at the point where the thick part of the black line meets the thin part of the blue line, are equal to one another, and the exterior angle at the point where the thin part of the black line meets the thin part of the red line, equal to the interior and opposite



angle upon the same side, viz the angle at the point where the thick part of the black line meets the thin part of the blue line, and the two interior angles upon the same side, viz the angle at the point where the thin part of the red line meets the thick part of the black line, and the angle at the point where the thick part of the black line meets the thin part of the blue line, together equal to two right angles

For if the angle at the point where the thick part of the red line meets the thick part of the black line be not equal to the angle at the point where the thick part of the black line meets the thin part of the blue line, one of them must be greater than the other We will suppose the angle at the point where the thick part of the red line meets the thick part of the black line to be the greater Then because the angle at the point where the thick part of the red line meets the thick part of the black line is greater than the angle at the point where the thick part of the black line meets the thin part of the blue line, to each of these add the angle at the point where the thin part of the red line meets the thick part of the black line Then according to Axiom IV, which says that if equals be added to unequals the wholes are unequal, the angle at the point where the thick part of the red line meets the thick part of the black line, and the angle at the point where the thin part of the red line meets the thick part of the black line, are together greater than the angle at the point where the thin part of the red line meets the thick part of the black line, and the angle at the point where the thick part of the black line meets the thin part of the blue line But, as shown in Proposition XIII, the angle at the point where the thick part of the red line meets the thick part of the black line, and the angle at the point where the thin part of the red line meets the thick part of the black line, are together equal to two right angles, therefore the angle at the point where

the thin part of the red line meets the thick part of the black line, and the angle at the point where the thick part of the black line meets the thin part of the blue line, are together less than two right angles But, according to Axiom XII, those straight lines which, with another straight line falling upon them, make the interior angles on the same side less than two right angles will meet together if continually lengthened, therefore the straight lines, viz the red line and the blue line, if lengthened far enough, will meet But they never meet because they are parallel (by the Hypothesis), therefore the angle at the point where the thick part of the red line meets the thick part of the black line is not unequal to the angle at the point where the thick part of the black line meets the thin part of the blue line, i e it is equal to it But, as shown in Proposition XV, the angle at the point where the thick part of the red line meets the thick part of the black line is equal to the angle at the point where the thin part of the black line meets the thin part of the red line, therefore, according to Axiom I, which says that things which are equal to the same thing are equal to one another, the angle at the point where the thin part of the black line meets the thin part of the red line is equal to the angle at the point where the thick part of the black line meets the thin part of the blue line If we now add to each of these equals the angle at the point where the thin part of the red line meets the thick part of the black line, it follows, according to Axiom II, which says that if equals be added to equals the wholes are equal, that the angle at the point where the thin part of the black line meets the thin part of the red line, and the angle at the point where the thin part of the red line meets the thick part of the black line, are equal to the angle at the point where the thin part of the red line meets the thick part of the black line, and the angle at the point where the thick part of the black line meets the thin part of the blue

line But, as shown in Proposition XIII, the angle at the point where the thin part of the black line meets the thin part of the red line, and the angle at the point where the thin part of the red line meets the thick part of the black line, are together equal to two right angles, therefore, according to Axiom I, which says that things which are equal to the same thing are equal to one another, the angle at the point where the thin part of the red line meets the thick part of the black line, and the angle at the point where the thick part of the black line meets the thin part of the blue line, are also together equal to two right angles We have thus proved what we had to prove

PROPOSITION XXX—THEOREM

Straight lines which are parallel to the same straight line are parallel to each other

First, we draw a straight line

Second, we draw two other straight lines, each of them

parallel to the line we first drew We have to prove that the blue line is parallel to the red line

Third, we draw a straight yellow line, cutting the blue line, the black line, and the red line

Then because the straight yellow line cuts the parallel straight lines, the blue line, and the black line, we know

that, as shown in Proposition XXIX, the angle at the point where the thick part of the blue line meets the thin part of the yellow line, is equal to the angle at the point where the thin part of the yellow line meets the thin part of the black line

Again, because the straight yellow line cuts the parallel straight lines, the black line, and the red line, we know that, as shown in Proposition XXIX, the angle at the point where the thin part of the yellow line meets the thin part of the black line is equal to the angle at the point where the dotted part of the yellow line meets the thin part of the red line And we have shown that the angle at the point where the thick part of the blue line meets the thin part of the yellow line is equal to the angle at the point where the thin part of the yellow line meets the thin part of the black line, therefore, according to Axiom I, which says that things which are equal to the same thing are equal to one another, the angle at the point where the thick part of the blue line meets the thin part of the yellow line is equal to the angle at the point where the dotted part of the yellow line meets the thin part of the red line And these two angles are alternate angles, therefore, as shown in Proposition XXVII, the blue line is parallel to the red line which is what we had to prove

Proposition XXXI —Problem

To draw a straight line through a given point parallel to a given straight line

First, we mark a point and draw a straight line We have to draw a line parallel to the black line, through the blue point

Second, in the black line we take any point, and join that point with the blue point

Third, at the blue point in the blue line, as shown in Proposition XXIII., we make the angle formed by the meeting of the blue line and the red line equal to the angle at

the point where the blue line meets the thin part of the black line.

Fourth, we produce (lengthen) the red line to any distance.

Then the red line shall be parallel to the black line, and we have to prove it.

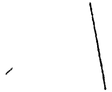
Because the straight blue line which meets the two straight lines, the red line and the black line, makes the alternate angles equal, viz. the angle at the point where the thick part of the red line meets the blue line, equal to the angle at the point where the blue line meets the thin part of the black line—and they are equal (for we made them so)—we know, from what we have shown in Proposition XXVII., that the red line is parallel to the black line. And the red line is drawn through the blue point: which is what we had to do.

Proposition XXXII.—Theorem

If a side of any triangle be produced (lengthened), the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are equal to two right angles.

First, we draw a triangle.

Second, we produce the blue side to any distance. Then we have to prove that the exterior angle, at the point where the black line meets the dotted blue line, is equal to the two interior and opposite angles, viz. the angle at the point where the black line meets the red line, and the angle at the point where the red line meets the blue line; and that the three interior angles of the triangle, viz. the angle at the point where the red line meets the blue line, the angle at the point where the blue line meets the black line, and the angle at the point where the black line meets the red line, are together equal to two right angles.



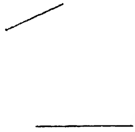
Third, through the point where the black line meets the dotted blue line, as shown in Proposition XXXI., we draw the dotted black line parallel to the red line.

Then, because the red line is parallel to the dotted black line, and the black line meets them, we know that, as shown in Proposition XXIX., the alternate angles are equal, viz. the angle at the point where the red line meets the black line, equal to the angle at the point where the black line meets the dotted black

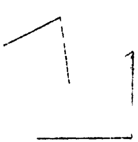
line Again, because the red line is parallel to the dotted black line and the blue line falls on them, we know that as shown in Proposition XXIX, the exterior angle, at the point where the dotted black line meets the dotted blue line, is equal to the interior and opposite angle, at the point where the red line meets the blue line But we have shown that the angle at the point where the black line meets the dotted black line, is equal to the angle at the point where the red line meets the black line, therefore, according to Axiom II, which says that if equals be added to equals the wholes are equal, the whole angle at the point where the black line meets the dotted blue line is equal to the two interior and opposite angles, at the point where the black line meets the red line, and at the point where the blue line meets the red line If we now add to these equals the angle at the point where the black line meets the blue line, it follows, according to Axiom II just mentioned, that the angle at the point where the black line meets the dotted blue line, and the angle at the point where the black line meets the blue line, are together equal to the angle at the point where the black line meets the red line, the angle at the point where the red line meets the blue line, and the angle at the point where the blue line meets the black line But, as shown in Proposition XIII, the angle at the point where the black line meets the dotted blue line, and the angle at the point where the black line meets the blue line, are together equal to two right angles, therefore, according to Axiom I, which says that things which are equal to the same thing are equal to one another, the three angles, viz the angle at the point where the black line meets the red line, the angle at the point where the red line meets the blue line, and the angle at the point where the blue line meets the black line, are together equal to two right angles We have thus proved what we had to prove

Corollary 1 (i e another fact proved) —From what we

have just proved, it follows that all the interior angles of any rectilineal figure (i e figure formed by straight lines), together with four right angles, are equal to twice as many right angles as the figure has sides



To prove this, let us draw any rectilineal figure Then by drawing straight lines to each of its angles from any point within the figure, we can divide it into as many triangles as the figure has sides And by the last proposition proved, we know that all the angles of these triangles are equal to twice as many right angles as there are triangles, as there are sides to the figure And the same angles are equal to the angles of the figure, together with the angles at the blue point, which is the vertex (top point) of each of the triangles And by Corollary 2 of Proposition XV we know that all the angles at the blue point are together equal to four right angles Therefore all the angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has sides



Corollary 2 (i e another fact proved) —It also follows from what we have just proved, that all the exterior angles of any rectilineal figure are together equal to four right angles

To prove this, let us draw any rectilineal figure

Then, because, as shown in Proposition XIII, every interior angle, as, for instance, the angle at the point where the blue line meets the black line, together with its adja-

cent exterior angle, viz the angle at the point where the blue line meets the dotted black line, is equal to two right angles, therefore all the interior angles together with all the exterior angles of the figure are equal to twice as many right angles as there are sides of the figure, i e as shown in the previous Corollary, they are equal to all the interior angles of the figure, together with four right angles therefore all the exterior angles are equal to four right angles

Proposition XXXIII —Theorem

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts are also themselves equal and parallel

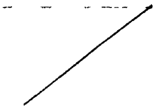
First, we draw two equal and parallel straight lines, viz the blue line and the red line

Second, we join them towards the same parts by straight lines, viz the dotted blue line and the dotted red line We have to prove that the dotted blue line and the dotted red line are equal and parallel

Third, we join the point where the dotted red line and the blue line meet, with the point where the red line and the dotted blue line meet

Then, because the red line is parallel to the blue line, and the black line meets them, the alternate angles are equal, as shown in Proposition XXIX, viz the angle at the point where the red line meets the black line, equal to the

angle at the point where the black line meets the blue line And because the red line is equal to the blue line (by



the Hypothesis), and the black line is part of each of the triangles, the one of which is formed by the red line, the black line, and the dotted red line, and the other by the black line, the blue line, and the dotted blue line, the two sides, viz the red side and the black side, are equal to the two sides, viz the black side and the blue side, each to each, and we have proved that the angle at the point where the red line meets the black line is equal to the angle at the point where the black line meets the blue line, therefore, as shown in Proposition IV, the base, viz the dotted red line, is equal to the base, viz the dotted blue line, and the triangle formed by the red line, the black line, and the dotted red line, equal to the triangle formed by the black line, the blue line, and the dotted blue line, and the other angles equal to the other angles, each to each, to which the equal sides are opposite therefore the angle at the point where the dotted red line meets the black line is equal to the angle at the point where the black line meets the dotted blue line And, as shown in Proposition XXVII, because the straight black line meets the two straight lines, the dotted red line and the dotted blue line, and makes the alternate angles, viz the angle at the point where the dotted red line meets the black line, and the angle at the point where the black line meets the dotted blue line, equal to one another, therefore the dotted red line is parallel to the dotted blue line And we have shown that it is equal to it so that we have proved what we had to prove

Proposition XXXIV —Theorem

The opposite sides and angles of parallelograms are equal to one another, and the diameter bisects them, i e divides them into two equal parts

N B -A parallelogram is a four sided figure of which the opposite sides are parallel, and the diameter is the straight line joining two of its opposite angles

First, we draw a parallelogram, of which the dotted red line is a diameter Then we have to show that the opposite sides and angles of the figure are equal to one another, and that the dotted red line divides it into two equal parts

Because the red line is parallel to the black line, and the dotted red line meets them, we know by Proposition XXIX that the alternate angles, viz the angle at the point where the red line meets the dotted red line, and the angle at the point where the dotted red line meets the black line, are equal And because the blue line is parallel to the yellow line, and the dotted red line meets them, it follows, by Proposition XXIX, that the alternate angles, viz the angle at the point where the blue line meets the dotted red line, and the angle at the point where the dotted red line meets the yellow line, are equal Therefore the triangle formed by the red line, the dotted red line, and the blue line, and the triangle formed by the dotted red line, the black line, and the yellow line have two angles in the one, viz the angle at the point where the red line meets the dotted red line, and the angle at the point where the dotted red line meets the blue line, equal to two angles in the other, viz the angle at the point where the dotted red line meets the black line, and the angle at the point where the dotted red

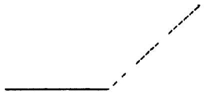
line meets the yellow line, each to each, and one side, viz the dotted red side, part of each of the triangles, and this side is next to their equal angles, therefore, as shown in Proposition XXVI, their other sides are equal, each to each, and the third angle of the one equal to the third angle of the other, viz the red side equal to the black side, and the blue side equal to the yellow side, and the angle at the point where the red line meets the blue line equal to the angle at the point where the yellow line meets the black line And because the angle at the point where the red line meets the dotted red line is equal to the angle at the point where the dotted red line meets the black line, and the angle at the point where the dotted red line meets the yellow line is equal to the angle at the point where the dotted red line meets the blue line, therefore, according to Axiom II, which says that if equals be added to equals the wholes are equal, the whole angle at the point where the red line meets the yellow line is equal to the whole angle at the point where the blue line meets the black line And we have shown that the angle at the point where the red line meets the blue line is equal to the angle at the point where the yellow line meets the black line, therefore the opposite sides and angles of parallelograms are equal to one another, which is the first part of what we had to prove Also the diameter of parallelograms divides them into two equal parts, for the red line being equal to the black line, and the dotted red line being the same in each case, the red line and the dotted red line are equal to the black line and the dotted red line, each to each And the angle at the point where the red line meets the dotted red line has been shown to be equal to the angle at the point where the dotted red line meets the black line, therefore, as shown in Proposition IV, the triangle formed by the red line, the dotted red line, and the blue line is equal to the triangle formed by the yellow line, the dotted red line, and the

black line, and consequently the dotted red line divides the parallelogram formed by the red, the blue, the black, and the yellow lines into two equal parts which is what we had to prove

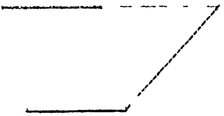
Proposition XXXV —Theorem

Parallelograms upon the same base and between the same parallels are equal to one another

First, let us draw two parallelograms upon the same base, viz the black line, and between the same parallels, viz the whole red line and the black line Then the parallelogram formed by the thick part of the red line the yellow line, the black line, and the blue line, shall be equal to the parallelogram formed by the thin part of the red line, the dotted black line, the black line, and the dotted blue line



If the thick red side of the one parallelogram and the thin red side of the other parallelogram, opposite to the base, viz the black line, meet and end at a point, it is plain, by what we proved in Proposition XXXIV, that each of the parallelograms is double of the triangle formed by the dotted blue line, the yellow line, and the black line, and that they are therefore equal, according to Axiom VI, which says that things which are double of the same are equal to one another

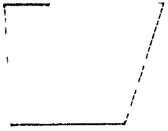


But let us draw two parallelograms in which the sides opposite the base do not meet and end in a point

Then, because the figure formed by the blue line, the black line, the yellow line, and the thick part of the red line, is a parallelogram, the thick part of the red line is equal to the black line, as shown in Proposition XXXIV, and for a similar reason the thin part of the red line is equal to the black line Consequently, according to Axiom I, which says that things which are equal to the same thing are equal to one another, the thick part of the red line is equal to the thin part of the red line And the dotted part of the red line is the same in each case, therefore the whole, viz the thick part of the red line and the dotted part, is equal to the whole, viz the dotted part of the red line and the thin part, according to Axiom II, which says that if equals be added to equals, the wholes are equal The blue line also we know is equal to the yellow line by what we proved in Proposition XXXIV, therefore the two, viz the line composed of the thick part of the red line and the dotted part, and the blue line, are equal to the two, viz the line composed of the thin part of the red line and the dotted part, and the yellow line, each to each And, as shown in Proposition XXIX, the exterior angle at the point where the dotted part of the red line meets the yellow line is equal to the interior angle at the point where the thick part of the red line meets the blue line, therefore, as shown in Proposition IV, the base, viz the dotted blue line, is equal to the base, viz the dotted black line, and the triangle formed by the thick and dotted parts of the red line, the blue line, and the dotted blue line, equal to the triangle formed by the thin and dotted parts of the red line, the yellow line, and the dotted black line If we now take these equal triangles in turn from the whole figure formed by the entire red line, the dotted black line, the black line, and the blue line, it follows, according to Axiom III, which says that if equals be taken from equals the remainders are equal, that the parallelogram formed by the thick part of the red line,

the blue line, the black line, and the yellow line, is equal to the parallelogram formed by the thin part of the red line, the dotted blue line, the black line, and the dotted black line which is what we had to prove

In the only other position in which the sides of the parallelograms opposite the black side can fall, viz when they overlap as in the figure we will now draw, the same proof holds good For by the same reasoning we can prove that the thick part of the red line and the dotted part are equal to the thin part and the dotted part If we then take away the dotted part, which is part of each of these equals, it follows, according to Axiom III, which says that if equals be taken from equals the remainders are equal, that the thick part of the red line is equal to the thin part of the red line By the same reasoning as in the previous position of the parallelograms we can then show that the triangle formed by the thick part of the red line, the blue line, and the dotted blue line, is equal to the triangle formed by the thin part of the red line the yellow line, and the dotted black line and by taking away these equals in turn from the whole figure, it follows as before that the remainders, viz the parallelograms, are equal We have therefore proved what we had to prove

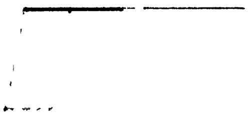


Proposition XXXVI—Theorem

Parallelograms upon equal bases and between the same parallels are equal to one another

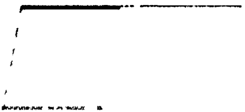
First, we draw two parallelograms upon equal bases, viz the thick red line and the thin red line, and between the

same parallels, viz the whole black line and the whole red line Then we have to prove that the parallelogram formed



by the thick black line, the thick yellow line, the thick red line, and the thick blue line is equal to the parallelogram formed by the thin black line, the thin yellow line, the thin red line, and the thin blue line

Second, we join the point where the thick blue line and the thick red line meet, with the point where the thin blue line and the thin black line meet, and the point where the thick red line and the thick yellow line meet, with the point where the thin black line and the thin yellow line meet



Then, because the thick red line is equal to the thin red line (by the Hypothesis), and the thin red line equal to the thin black line (by what we proved in Proposition XXXIV), therefore, according to Axiom I, which says that things which are equal to the same thing are equal to one another, the thick red line is equal to the thin black line And, according to the Hypothesis, these two lines are parallel, and they are joined towards the same parts by the straight dotted blue line and the straight dotted yellow line But we showed in Proposition XXXIII that straight lines which join the ends of equal and parallel straight lines towards the same parts are themselves equal and parallel therefore the dotted blue line and the dotted yellow line are both equal and parallel Consequently, according to the definition at the commencement of Proposition XXXIV, which says that a parallelogram is a four-sided figure of which the opposite sides are parallel, the figure

formed by the thick red line, the dotted yellow line, the thin black line, and the dotted blue line, is a parallelogram And according to Proposition XXXV it is equal to the parallelogram formed by the thick red line, the thick yellow line, the thick black line, and the thick blue line, because it is upon the same base, viz the thick red line, and between the same parallels, viz the thick red line and the whole black line For the like reason the parallelogram formed by the thin red line, the thin yellow line, the thin black line, and the thin blue line, is also equal to the parallelogram formed by the thick red line, the dotted yellow line, the thin black line, and the dotted blue line Therefore, according to Axiom I, which says that things which are equal to the same thing are equal to one another, the parallelogram formed by the thick red line, the thick yellow line, the thick black line, and the thick blue line, is equal to the parallelogram formed by the thin red line, the thin yellow line, the thin black line, and the thin blue line which is what we had to prove

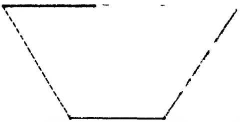
Proposition XXXVII —Theorem

Triangles upon the same base, and between the same parallels, are equal to one another

First, we draw two triangles, viz the triangle formed by the thin blue line, the yellow line, and the black line, and the triangle formed by the dotted blue line, the dotted yellow line, and the black line, upon the same base, viz the black line, and between the same parallels, viz the thin red line and the black line then we have to prove that the two triangles are equal

Second, as allowed by Postulate II, which says that a straight line may be produced to any length in a straight

line, we lengthen the thin red line both ways, and, as shown in Proposition XXXI., through the point where the blue



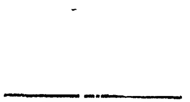
line and the black line meet, we draw the dotted black line parallel to the yellow line, and through the point where the dotted yellow line and the black line meet, we draw the thick blue line parallel to the dotted blue line.

Then, according to the Definition to Proposition XXXIV., the figure formed by the thick part of the red line, the dotted black line, the black line, and the yellow line, is a parallelogram, and the figure formed by the dotted blue line, the black line, the thick blue line, and the dotted part of the red line, is also a parallelogram, and these two parallelograms are equal, as shown in Proposition XXXV., because they are on the same base, viz. the black line, and between the same parallels, viz. the black line and the whole red line. And the triangle formed by the thin blue line, the black line, and the yellow line, is the half of the parallelogram formed by the thick part of the red line, the dotted black line, the black line, and the yellow line, because, as shown in Proposition XXXIV., the diameter of the parallelogram, viz. the thin blue line, divides it into halves. In the same way the triangle formed by the dotted blue line, the black line, and the dotted yellow line, is the half of the parallelogram formed by the dotted part of the red line, the dotted blue line, the black line, and the thick blue line, because the diameter of the parallelogram, viz. the dotted yellow line, divides it into halves. But according to Axiom VII. the halves of equal things are equal: therefore the triangle formed by the thin blue line, the yellow line, and the black line, is equal to the triangle formed by the dotted blue line, the dotted yellow line, and the black line: which is what we had to prove.

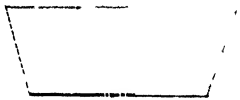
Proposition XXXVIII —Theorem

Triangles upon equal bases and between the same parallels are equal to one another

First, we draw two triangles, viz the triangle formed by the thin blue line, the yellow line, and the thick part of the black line, and the triangle formed by the dotted blue line, the dotted yellow line, and the thin part of the black line, upon equal bases, viz the thick part of the black line and the thin part of the black line, and between the same parallels, viz the red line and the whole black line Then we have to prove that these two triangles are equal



Second, as allowed by Postulate II, which says that a straight line may be produced to any length, we lengthen the thin red line both ways, and through the point where the thin blue line meets the thick part of the black line we draw the dotted black line parallel to the yellow line, as shown in Proposition XXXI, and through the point where the dotted yellow line meets the thin part of the black line, we draw the thick blue line parallel to the dotted blue line Then, according to the Definition to Proposition XXXIV, the figure formed by the thick part of the red line, the dotted black line, the thick part of the black line, and the yellow line, is a parallelogram, and the figure formed by the dotted part of the red line, the dotted blue line, the thin part of the black line, and the thick blue line, is a parallelogram And, as shown in Proposition XXXVI, these two parallelograms are equal to one another because they are upon equal bases, viz. the thick part of the black



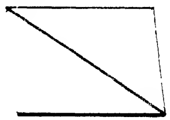
line and the thin part of the black line, and between the same parallels, viz the whole black line and the red line And, as shown in Proposition XXXIV, the triangle formed by the thin blue line, the thick part of the black line, and the yellow line, is the half of the parallelogram formed by the thick part of the red line, the dotted black line, the thick part of the black line, and the yellow line, because the diameter of the parallelogram, viz the thin blue line, divides it into halves And for the same reason the triangle formed by the dotted blue line, the thin part of the black line, and the dotted yellow line, is the half of the parallelogram formed by the dotted part of the red line, the dotted blue line, the thin part of the black line, and the thick blue line, because the diameter of the parallelogram, viz the dotted yellow line, divides it into halves But, according to Axiom VII, the halves of equal things are equal, therefore the triangle formed by the thin blue line, the thick part of the black line, and the yellow line, is equal to the triangle formed by the dotted blue line, the thin part of the black line, and the dotted yellow line which is what we had to prove

Proposition XXXIX —Theorem

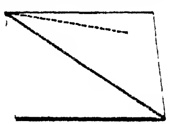
Equal triangles upon the same base, and upon the same side of it, are between the same parallels

First, we draw two equal triangles, viz the triangle formed by the thick black line, the thick blue line, and the thick red line, and the triangle formed by the thick black line, the thin blue line, and the thin red line, upon the same base, viz the thick black line, and upon the same side of it We have to prove they are between the same parallels.

Second, we join the point where the thick red line and the thick blue line meet with the point where the thin red line and the thin blue line meet Then the thin black line shall be parallel to the thick black line



For if it be not, then, as shown in Proposition XXXI, through the point where the thick blue line and the thick red line meet, draw the dotted black line parallel to the thick black line, and join the point where the dotted black line cuts the thin blue line with the point where the thick black line meets the thin red line



Then the triangle formed by the thick red line, the thick blue line, and the thick black line, is equal to the triangle formed by the dotted blue line, the part of the thin blue line joining the dotted blue line with the thick black line, and the thick black line, as shown in Proposition XXXVII, because they are upon the same base, viz the thick black line, and between the same parallels, viz the thick black line and the dotted black line But according to the Hypothesis the triangle formed by the thick red line, the thick blue line, and the thick black line, is equal to the triangle formed by the thin red line, the thin blue line, and the thick black line therefore, according to Axiom I, which says that things which are equal to the same thing are equal to one another, the triangle formed by the thin red line, the thin blue line, and the thick black line is equal to the triangle formed by the dotted blue line, the part of the thin blue line joining the dotted blue line with the thick black line, and the thick black line, that is to say, the greater equal to the less, which is impossible therefore the dotted black line is not parallel to the thick black line. In the same

way it can be shown that no other line but the thin black line is parallel to the thick black line therefore the thin black line is parallel to the thick black line which is what we had to prove

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Proposition XL —Theorem

Equal triangles upon equal bases in the same straight line and towards the same parts are between the same parallels

First, we draw two equal triangles, viz the triangle formed by the thick black line, the thick blue line, and the thick red line, and the triangle formed by the thin black line, the thin blue line, and the thin red line, upon equal bases, viz the thick part of the black line and the thin part of the black line, in the same straight line, viz the whole black line, and towards the same parts We have to prove they are between the same parallels

Second, we join the point where the thick blue line and the thick red line meet with the point where the thin blue line and the thin red line meet then the dotted blue line shall be parallel to the whole black line

For if it be not, through the point where the thick blue line and the thick red line meet, draw, as shown in Proposition XXXI, the dotted yellow line parallel to the whole black line, and join the point where the dotted yellow line meets the

thin blue line with the point where the thin black line meets the thin red line Then, as shown in Proposition XXXVIII, the triangle formed by the thick red line, the thick blue line, and the thick black line, is equal to the triangle formed by the thin yellow line, the part of the thin blue line between the thin yellow line and the thin black line, and the thin black line, because they are upon equal bases, viz the thick part of the black line and the thin part of the black line, and between the same parallels, viz the dotted yellow line and the whole black line But by the Hypothesis the triangle formed by the thick red line, the thick blue line, and the thick black line, is equal to the triangle formed by the thin red line, the thin blue line, and the thin black line Therefore, according to Axiom I, which says that things which are equal to the same thing are equal to one another, the triangle formed by the thin red line, the thin blue line, and the thin black line, is equal to the triangle formed by the thin yellow line, the part of the thin blue line between the thin yellow line and the thin black line, and the thin black line i e the greater is equal to the less, which is impossible Therefore the dotted yellow line is not parallel to the whole black line In the same way it can be shown that no other line is parallel to the whole black line, but the dotted blue line therefore the dotted blue line is parallel to the whole black line which is what we had to prove

PROPOSITION XLI —THEOREM

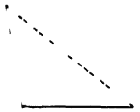
If a parallelogram and a triangle be upon the same base and between the same parallels, the parallelogram shall be double of the triangle

First, we draw the parallelogram formed by the black line, the red line, the yellow line, and the blue line, and the

triangle formed by the black line, the dotted red line, and the dotted blue line, upon the same base, viz the black line, and

between the same parallels, viz the black line and the whole yellow line Then the parallelogram formed by the black line, the red line, the yellow line, and the blue line, shall be the double of the triangle formed by the black line, the dotted red line, and the dotted blue line this we have to prove

Second, we join the point where the red line and the yellow line meet with the point where the black line and the blue line meet



Then the triangle formed by the black line, the red line, and the dotted black line, is equal to the triangle formed by the black line, the dotted red line, and the dotted blue line, as shown in Proposition XXXVII, because they are upon the same base, viz the black line, and between the same parallels, viz the black line and the whole yellow line But the parallelogram formed by the black line, the red line, the yellow line, and the blue line, is double of the triangle formed by the black line, the red line, and the dotted black line, because, as shown in Proposition XXXIV, the diameter, viz the dotted black line, divides the parallelogram into two equal parts therefore the parallelogram formed by the black line, the red line, the yellow line, and the blue line is also double of the triangle formed by the black line, the dotted red line, and the dotted blue line which is what we had to prove

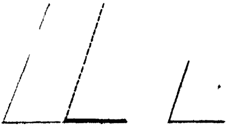
Proposition XLII.—Problem

To describe (make) a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

First, we draw a triangle and an angle. Then what we have to do is to draw a parallelogram equal to the triangle formed by the black line, the red line, and the blue line, and having one of its angles equal to the angle formed by the meeting of the two short black lines.

Second, as shown in Proposition X., we divide the black line into two equal parts, and join the point of division with the point where the red side and the blue side of the triangle meet.

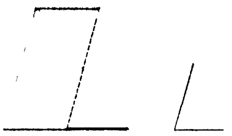
Third, as shown in Proposition XXIII., at the point where the yellow line meets the thick part of the black line, we make the angle formed by the dotted black line and the thick part of the black line equal to the angle formed by the meeting of the two short black lines.



Fourth, as shown in Proposition XXXI., through the point where the red line and the blue line meet, we draw the thick and dotted red line parallel to the thick black line.

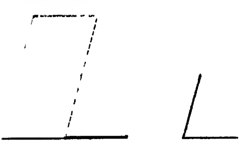
Fifth, in the same way, through the point where the black

line and the blue line meet, we draw the dotted blue line parallel to the dotted black line Then, according to the



Definition to Proposition XXXIV, the figure formed by the thick black line, the dotted black line, the dotted red line, and the dotted blue line, is a parallelogram

And because by the Hypothesis the thin part of the black line is equal to the thick part of the black line, we know by what we proved in



Proposition XXXVIII that the triangle formed by the thin part of the black line, the thin red line, and the yellow line, is equal to the triangle formed by the thick part of the black line, the yellow line, and the blue line, because they are upon equal bases, viz the thin part of the black line and the thick part of the black line, and between the same parallels, viz the whole black line and the thick and dotted red line therefore the triangle formed by the whole black line, the thin red line, and the blue line, is double of the triangle formed by the thick part of the black line, the yellow line, and the blue line But, as shown in Proposition XLI, the parallelogram formed by the thick part of the black line, the dotted black line, the dotted red line, and the dotted blue line, is also double of the triangle formed by the thick part of the black line, the yellow line, and the blue line, because they are upon the same base, viz the thick part of the black line, and between the same parallels, viz the thick part of the black line and the thick and dotted red line Therefore, according to Axiom VI, which says that things which are the double of the same are equal to

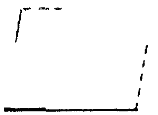
one another, the parallelogram formed by the thick part of the black line, the dotted black line, the dotted red line, and the dotted blue line, is equal to the triangle formed by the whole black line, the thin red line and the blue line And it has one of its angles, viz the angle at the point where the dotted black line meets the thick part of the black line, equal to the angle formed by the meeting of the two short black lines (for we made it so) therefore we have made the parallelogram we had to make

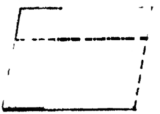
Proposition XLIII —Theorem

The complements of the parallelograms (i e the four-sided figures cut off by straight lines within the parallelograms) which are about the diameter of any parallelogram are equal to one another

First, we draw a parallelogram of which the dotted blue line is the diameter, i e divides it into equal parts

Second, we draw two smaller parallelograms about the dotted blue line, i e through which the dotted blue line passes, viz the parallelogram formed by the thick part of the red line, the thick part of the blue line, the thick part of the yellow line, and the dotted black line, and the parallelogram formed by the dot-and dash black line, the thin part of the yellow line, the thin part of the black line, and the dot and-dash red line





Then the parallelogram formed by the thin part of the red line, the thick part of the black line, the thin

part of the yellow line, and the dotted black line, and the parallelogram formed by the thick part of the yellow line, the dot-and-dash black line, the dotted red line, and the thin part of the blue line, are the other parallelograms which make up the whole figure first drawn, and which are therefore called the complements Then these complements shall be equal

Because the whole figure formed by the whole thick and thin black line, the whole thick and thin red line, the whole thick and thin blue line, and the whole dotted and dot-and-dash red line, is a parallelogram, and the dotted blue line is its diameter, we know by Proposition XXXIV that the triangle formed by the whole thick and thin red line, the whole thick and thin black line, and the dotted blue line, is equal to the triangle formed by the whole thick and thin blue line, the whole dotted and dot and-dash red line, and the dotted blue line Again, because the figure formed by the thick part of the blue line, the thick part of the red line, the dotted black line, and the thick part of the yellow line, is a parallelogram, the diameter of which is the part of the dotted blue line, therefore, as shown in Proposition XXXIV, the triangle formed by the thick part of the red line, the dotted black line, and the part of the dotted blue line, is equal to the triangle formed by the thick part of the blue line, the thick part of the yellow line, and the part of the dotted blue line And for the same reason the triangle formed by the thin part of the yellow line, the thin part of the black line, and the other part of the dotted blue line, is equal to the triangle formed by the dot-and-dash black line, the dot-and dash red line, and the other part of the dotted blue line Therefore, because the triangle formed by the thick part of the red line, the dotted black line, and the part of the dotted blue line, is equal to the triangle formed by the thick part of the blue line, the thick part of the yellow line, and the part of the dotted blue line, and the

triangle formed by the thin part of the yellow line, the thin part of the black line, and the other part of the dotted blue line, is equal to the triangle formed by the dot and dash black line, the dot and-dash red line, and the other part of the dotted blue line, therefore the triangle formed by the thick part of the red line, the dotted black line, and the part of the dotted blue line, together with the triangle formed by the thin part of the yellow line, the thin part of the black line, and the other part of the dotted blue line, are equal to the triangle formed by the thick part of the blue line, the thick part of the yellow line, and the part of the dotted blue line, together with the triangle formed by the dot-and dash black line, the dot and dash red line, and the other part of the dotted blue line, according to Axiom II, which says that if equals be added to equals, the wholes are equal But the whole triangle formed by the whole thick and thin red line, the whole thick and thin black line, and the whole dotted blue line, was shown to be equal to the whole triangle formed by the whole thick and thin blue line, the whole dotted and dot-and-dash red line, and the whole dotted blue line, therefore, according to Axiom III, which says that if equals be taken from equals, the remainders are equal, the remaining complement, viz the figure formed by the thin part of the red line, the thick part of the black line, the thin part of the yellow line, and the dotted black line, is equal to the remaining complement, viz the thin part of the blue line, the dotted red line, the dot and-dash black line, and the thick part of the yellow line which is what we had to prove

Proposition XLIV.—Problem.

To a given straight line to apply a parallelogram which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

First, we draw the straight line to which the parallelogram is to be applied, and the triangle and the angle.

What we have to do is to apply to the thick red line a parallelogram which shall be equal to the triangle formed by the three black lines, and shall have an angle equal to the angle contained by the two blue lines.

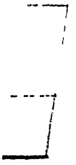
Second, as shown in Proposition XLII., we make the parallelogram formed by the thin red line, the thick blue line, the thin yellow line, and the dotted black line, equal to the triangle formed by the three black lines, and containing the angle at the point where the dotted black line meets the thin red line, equal to the angle contained by the two blue lines, so that the thin red line is in the same straight line with the thick red line.

Third, we lengthen the thin yellow line, and, as shown in Proposition XXXI., through the lower end of the thick red line we draw the thick black line meeting the lengthened yellow line, and parallel either to the dotted black line or to the thick blue line.

Fourth, we join the point where the thick yellow line

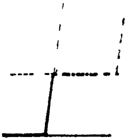
meets the thick black line with the point where the dotted black line meets the thick red line

Then because the whole straight yellow line falls upon the parallels, viz the thick black line and the thick blue line, it follows, according to what we showed in Proposition XXIX, that the angle at the point where the thick black line meets the thick yellow line, and the angle at the point where the thin yellow line meets the thick blue line, are together equal to two right angles Therefore the angle at the point where the dotted red line meets the thick yellow line, and the angle at the point where the thin yellow line meets the thick blue line, are together less than two right angles But, according to Axiom XII, straight lines which with another straight line make the interior angles upon the same side less than two right angles, meet if lengthened far enough therefore the dotted red line and the thick blue line will meet if lengthened far enough



Fifth, let us lengthen them until they meet, and through the point where they meet, draw the dot-and-dash and dotted blue line parallel either to the whole red line or to the whole yellow line

Sixth, we lengthen the dotted black line and the thick black line to meet the dot-and-dash and dotted blue line



Then the whole figure formed by the whole thick and thin black line, the whole dotted and dot-and-dash blue line, the whole thick and

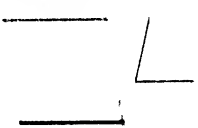
thin blue line, and the whole thick and thin yellow line, is a parallelogram, of which the diameter is the whole dotted red line, and the parallelogram formed by the thick black line, the thick yellow line, the dotted black line, and the thick red line, and the parallelogram formed by the dot-and-dash black line, the thin red line, the thin blue line, and the dot and-dash blue line, are parallelograms about the diameter, i e through which the diameter passes And the parallelogram formed by the thin black line, the dotted blue line, the dot-and-dash black line, and the thick red line, and the parallelogram formed by the dotted black line, the thin red line, the thick blue line, and the thin yellow line, are the complements of the two parallelograms just mentioned, through which the diameter passes, and therefore, as shown in Proposition XLIII, they are equal to one another But we made the parallelogram formed by the dotted black line, the thin yellow line, the thick blue line, and the thin red line equal to the triangle formed by the three black lines therefore, according to Axiom I, which says that things which are equal to the same thing are equal to one another, the parallelogram formed by the thin black line, the dotted blue line, the dot-and-dash black line, and the thick red line, is also equal to the triangle formed by the three black lines And because the angle at the point where the dotted black line meets the thin red line is equal to the angle at the point where the thick red line meets the dot and-dash black line (as shown in Proposition XV), and also to the angle formed by the meeting of the two blue lines, to which we had to make one angle in the required parallelogram equal (for we made it so), therefore, according to Axiom I, which says that things which are equal to the same thing are equal to one another, the angle at the point where the thick red line meets the dot-and-dash black line is also equal to the angle formed by the meeting of the two blue lines. We have therefore done what was required,

viz applied to the straight thick red line the parallelogram formed by the thick red line, the dot-and-dash black line, the dotted blue line, and the thin black line, equal to the triangle formed by the three black lines, and having the angle at the point where the thick red line meets the dot-and dash black line equal to the angle formed by the meeting of the two blue lines

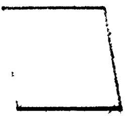
Proposition XLV —Problem

To describe (make) a parallelogram equal to a given rectilineal figure and having an angle equal to a given rectilineal angle

First, we draw the given rectilineal figure (i e figure formed by straight lines), and also the given angle What we have to do is to make a parallelogram equal to the figure formed by the thick black line, the thick blue line, the thick red line, and the thick yellow line, and having an angle equal to the angle formed by the meeting of the two black lines



First, we join the point where the black side meets the blue side of the given figure with the point where the red side meets the yellow side of the figure.



Second, as shown in Proposition XLII, we make the parallelogram formed by the thin blue line, the thin black line, the thin yellow line, and the thin red line, equal to the triangle formed by the thick blue line, the thick red line, and the dotted blue line,

and having the angle at the point where the thin blue line meets the thin black line, equal to the angle formed by the meeting of the two black lines

Third, as shown in Proposition XLIV, we apply to the thin yellow line the parallelogram formed by the thin yellow line, the dotted red line, the dotted yellow line, and the dotted black line, equal to the triangle formed by the thick yellow line, the thick black line, and the dotted blue line, and having the angle at the point where the thin yellow line meets the dotted black line, equal to the angle formed by the meeting of the two black lines The figure formed by the whole thin black and dotted black line, the thin blue line, the whole thin red and dotted red line, and the dotted yellow line, shall be the parallelogram required

Because the angle formed by the meeting of the two black lines is equal to the angle at the point where the thin blue line meets the thin black line, and also to the angle at the point where the thin yellow line meets the dotted black line (for we made it so), therefore, according to Axiom I, which says that things which are equal to the same thing are equal to one another, the angle at the point where the thin blue line meets the thin black line is equal to the angle at the point where the thin yellow line meets the dotted black line Add to each of these the angle at the point where the thin black line meets the thin yellow line, then, according to Axiom II, which says that if equals be added to equals the wholes are equal, the angle at the point where the thin blue line meets the thin black line, and the angle at the point where the thin black line meets the thin yellow line, are together equal to the angle at the point where the thin

black line meets the thin yellow line, and the angle at the point where the thin yellow line meets the dotted black line But we know, by what we proved in Proposition XXIX, that the angle at the point where the thin blue line meets the thin black line, and the angle at the point where the thin black line meets the thin yellow line, are together equal to two right angles, therefore, according to Axiom I, which says that things which are equal to the same thing are equal to one another, the angle at the point where the thin black line meets the thin yellow line, and the angle at the point where the thin yellow line meets the dotted black line, are also equal to two right angles And because at the point in the straight thin yellow line where it meets the thin black line, the two straight lines, viz the thin black line and the dotted black line, upon opposite sides of it make the adjacent angles equal to two right angles, we know, by what we showed in Proposition XIV, that the thin black line is in the same straight line with the dotted black line And because the straight thin yellow line meets the parallel straight lines, viz the whole thin black and dotted black line, and the thin red line, we know, as shown in Proposition XXIX, that the alternate angles, viz the angle at the point where the dotted black line meets the thin yellow line, and the angle at the point where the thin yellow line meets the thin red line, are equal If we now add to each of these the angle at the point where the thin yellow line meets the dotted red line, it follows, according to Axiom II, which says that if equals be added to equals the wholes are equal, that the angle at the point where the dotted black line meets the thin yellow line, and the angle at the point where the thin yellow line meets the dotted red line, are together equal to the angle at the point where the thin yellow line meets the thin red line, and the angle at the point where the thin yellow line meets the dotted red line But we know, by what we proved in

Proposition XXIX, that the angle at the point where the dotted black line meets the thin yellow line, and the angle at the point where the thin yellow line meets the dotted red line, are together equal to two right angles therefore, according to Axiom I, which says that things which are equal to the same thing are equal to one another, the angle at the point where the thin yellow line meets the thin red line, and the angle at the point where the thin yellow line meets the dotted red line, are also equal to two right angles, and consequently, according to what we proved in Proposition XIV, the thin red line is in the same straight line with the dotted red line And because the thin blue line is parallel to the thin yellow line, and the thin yellow line to the dotted yellow line (for we made them so), it follows, by what we proved in Proposition XXX, that the thin blue line is parallel to the dotted yellow line And the whole thin and dotted black line, and the whole thin and dotted red line, were made parallels therefore, according to the Definition to Proposition XXXIV, the figure formed by the thin blue line, the whole thin and dotted red line, the dotted yellow line, and the whole thin and dotted black line, is a parallelogram And because the triangle formed by the thick blue line, the thick red line, and the dotted blue line, is equal to the parallelogram formed by the thin black line, the thin blue line, the thin red line, and the thin yellow line (for we made it so), and the triangle formed by the thick black line, the thick yellow line, and the dotted blue line, is equal to the parallelogram formed by the dotted black line, the thin yellow line, the dotted red line. and the dotted yellow line (for we made it so), it follows, according to Axiom II., which says that if equals be added to equals the wholes are equal, that the whole figure formed by the thick black line, the thick blue line, the thick red line, and the thick yellow line, is equal to the parallelogram formed by the thin blue line, the whole red and dotted red line, the dotted yellow line,

and the whole thin and dotted black line And the angle at the point where the thin blue line meets the thin black line was made equal to the angle formed by the meeting of the two black lines, so that we have done what was required, viz made a parallelogram equal to the given rectilineal figure, and containing an angle equal to the stated angle

Corollary (i e another fact proved)*—From this it is plain how to apply to a straight line a parallelogram which shall have an angle equal to a given rectilineal angle and shall be equal to a given rectilineal figure, viz by applying to the straight line, as shown in Proposition XLIV, a parallelogram equal to the first triangle formed by the thick blue line, the thick red line, and the dotted blue line, and having an angle equal to the given angle

PROPOSITION XLVI —PROBLEM

To describe (make) a square upon a given straight line

First, we draw a straight line upon which to make a square

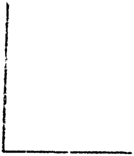
Second, as shown in Proposition XI, from either end of the black line we draw the blue line at right angles to the black line, and, as shown in Proposition III, make the thick part of the blue line equal to the black line

Third, as shown in Proposition XXXI, through the other end of the thick part of the blue line to that which meets the black line, we draw the red line parallel to the black line, and through the other end of the black line to that which meets the blue line, draw the yellow line parallel to the blue line

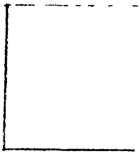
Then, according to the Definition to Proposition XXXIV, the figure formed by the black line, the thick blue line, the

red line, and the yellow line, is a parallelogram, and it follows, as shown in Proposition XXXIV, that the black line is

equal to the red line, and the thick blue line to the yellow line. But we made the thick blue line equal to the black line, therefore, according



to Axiom I, which says that things which are equal to the same thing are equal to one another, the four straight lines, viz the black line, the thick blue line, the red line, and the yellow line, are equal to one another, and the parallelogram formed by those four lines has, consequently, equal sides. Also all its angles are right angles, for since the straight thick blue line meets the parallel black line and red line, we know, by what we showed in Proposition XXIX, that the angle at the point where the black line meets the



thick blue line, and the angle at the point where the red line meets the thick blue line, are together equal to two right angles. But we made the angle at the point where the black line meets the thick blue line a right angle, therefore, according to Axiom III, which says that if equals be taken from equals the remainders are equal, the angle at the point where the red line meets the thick blue line is a right angle. But we know, by what we proved in Proposition XXXIV, that the opposite angles of parallelograms are equal; therefore, according to Axiom I, which says that things which are equal to the same thing are equal to one

another, each of the opposite angles, viz the angle at the point where the red line meets the yellow line, and the angle at the point where the black line meets the yellow line, is a right angle Consequently each of the angles of the figure formed by the black line, the thick blue line, the red line, and the yellow line, is a right angle, and it has been shown that the sides of the figure are all equal, consequently, according to Definition XXX, the figure is a square, and it has been made upon the straight black line which is what we had to do

Corollary (i e another fact proved)—From what we have just done, it is evident that every parallelogram which has one right angle has all its angles right angles

PROPOSITION XLVII—THEOREM

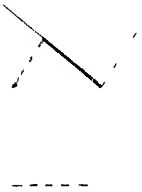
In any right-angled triangle, the square which is described (made) upon the side subtending (opposite to) the right angle is equal to the squares described upon the sides which contain the right angle

First, we draw a right-angled triangle, having the angle at the point where the thick black line meets the thick red line a right angle The square made upon the whole blue line shall be equal to the squares made upon the thick red line and the thick black line

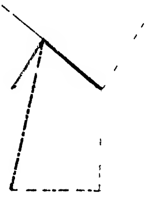
Second, as shown in Proposition XLVI, on the thick red line we make the square formed by the thick red line, the thin black line, the thin red line, and the thick yellow line, on the thick black line, the square formed by the thick black line, the dotted red line, the dotted blue line, and the dotted black line, and on the whole blue line the square formed

by the whole blue line, the dot-and-dash red line, the thin dot and dash black line, and the dot-and dash blue line

Third, as shown in Proposition XXXI, through the point where the thick red line and the thick black line meet, we draw the thin and dotted yellow line parallel either to the dot and-dash blue line or to the dot-and-dash red line, and join the point where the thick red line and the thick black line meet, with the point where the dot and dash blue line and the thin dot and dash black line meet, and the point where the thick yellow line and the thin red line meet, with the point where the thick black line meets the thin part of the blue line



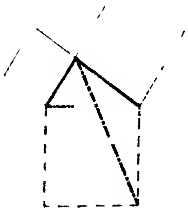
Then, because the angle at the point where the thick red line meets the thick black line is a right angle (for we made it so), and that the angle at the point where the thick red line meets the thin black line is a right angle, according to Definition XXX, which says that all the angles of a square are right angles, therefore the two straight lines, viz the thick black line and the thin black line, upon the opposite sides of the thick red line, make with it, at the point where they meet it, the adjacent angles equal to two right angles, and consequently, as shown in Proposition XIV, the thick black line is in the same straight line with the thin black line In the same way we can show that the thick red line and the dotted red line are in the



same straight line And because the angle at the point where the dot and dash blue line meets the thick blue line is equal to the angle at the point where the thick yellow line meets the thick red line, because each of them is a right angle, according to Definition XXX, which says that all the angles of a square are right angles, and that all right angles are equal, according to Axiom XI, add to each the angle at the point where the thick red line meets the thick blue line It follows, according to Axiom II, which says that if equals be added to equals the wholes are equal, that the whole angle at the point where the thick red line meets the dot-and-dash blue side is equal to the whole angle at the point where the thick yellow line meets the thick and thin blue line And because the thick red side and the dot-and-dash blue side are equal to the thick yellow side and the whole thick and thin blue side, each to each, according to Definition XXX, which says that all the sides of a square are equal, and the angle contained by the dot-and-dash blue side and the thick red side is equal to the angle contained by the thick yellow side and the thick and thin blue side, therefore, as shown in Proposition IV, the base, viz the thick dot-and dash black line, is equal to the base, viz. the dot-and dash yellow line, and the triangle formed by the thick red line, the dot-and-dash blue line, and the thick dot and-dash black line is equal to the triangle formed by the thick yellow line, the whole thick and thin blue line, and the dot-and-dash yellow line Now, according to what we proved in Proposition XLI, the parallelogram formed by the thick part of the blue line, the dotted yellow line, the dot-and-dash blue line, and the part of the thin dot-and-dash black line between the dot-and dash blue line and the dotted yellow line, is double the triangle formed by the thick red line, the dot-and dash blue line, and the thick dot-and-dash black line, because they are upon the same base, viz the dot-and dash blue line, and between the same parallels, viz. the

dot-and dash blue line and the whole thin and dotted yellow line, and the square formed by the thick red line, the thick yellow line, the thin red line, and the thin black line, is double the triangle formed by the thick yellow line, the dot-and-dash yellow line, and the whole thick and thin blue line, because they are on the same base, viz the thick yellow line, and between the same parallels, viz the thick yellow line and the whole thin and thick black line But, according to Axiom VI, the doubles of equals are equal to one another, therefore the parallelogram formed by the thick part of the blue line, the dot-and dash blue line, the dotted yellow line, and the part of the thin dot and-dash black line, is equal to the square formed by the thick red line, the thick yellow line, the thin red line, and the thin black line

In the same way, if we join the point where the thick red line and the thick black line meet, with the point where the dot-and dash red line and the thin dot and-dash black line meet, and the point where the thick red line and the thick blue line meet, with the point where the dotted blue line and the dotted black line meet, we can show that the parallelogram formed by the thin part of the blue line, the dotted yellow line, the dot-and-dash red line, and the other part of the thin dot-and-dash black line, is equal to the square formed by the thick black line, the dotted red line, the dotted blue line, and the dotted black line Consequently, according to Axiom II, which says that if equals be added to equals the wholes are equal, the whole square formed by the whole thick and thin blue line, the dot-and-dash red line, the thin dot-and-dash black line, and the dot and-dash blue line, is equal to the square formed by the thick red line, the thick



yellow line, the thin red line, and the thin black line, and the square formed by the thick black line, the dotted red line, the dotted blue line, and the dotted black line, or, in other words, the square made upon the blue line is equal to the squares made upon the thick red line and the thick black line which is what we had to prove

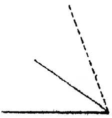
PROPOSITION XLVIII —THEOREM

If the square described (made) upon one of the sides of a triangle be equal to the squares described upon the other two sides of it, the angle contained by these two sides is a right angle

First, we draw a triangle, such that the square made on the black side is equal to the squares made on the blue and the red sides We have to prove that the angle at the point where the red side meets the blue side is a right angle

Second, as shown in Proposition XI, at the point where the red side meets the blue side we draw the dotted blue line at right angles to the red side, and, as shown in Proposition III, make the dotted blue line equal to the blue side

Third, we join the end of the dotted blue line with the point where the red side meets the black side



Then, because the dotted blue line is equal to the blue side, the square of the dotted blue line is equal to the square of the blue side To each of these add

the square of the red side then, according to Axiom II, which says that if equals be added to equals the wholes are equal, the squares of the dotted blue line and the red side are equal to the squares of the blue side and the red side But, as shown in Proposition XLVII, the square of the dotted black line is equal to the squares of the dotted blue line and the red side, because the angle at the point where the dotted blue line meets the red side is a right angle (for we made it so) And the square of the black side, according to the Hypothesis, is equal to the squares on the blue side and the red side therefore, according to Axiom I, which says that things which are equal to the same thing are equal to one another, the square of the dotted black line is equal to the square of the black side, and consequently the dotted black line is equal to the black side And because the dotted blue line is equal to the blue side (for we made it so), and the red side is part of each of the two triangles, the one of which is formed by the dotted blue line, the red side, and the dotted black line, and the other by the blue side, the red side, and the black side, the two sides, viz the dotted blue line and the red side, are equal to the two sides, viz the blue side and the red side, each to each, and the base, viz the dotted black line, has been shown to be equal to the base, viz the black side, therefore, as shown in Proposition VIII, the angle at the point where the dotted blue line meets the red side is equal to the angle at the point where the blue side meets the red side But we made the angle at the point where the dotted blue line meets the red side, a right angle, therefore, according to Axiom I, which says that things which are equal to the same thing are equal to one another, the angle at the point where the blue side meets the red side is a right angle which is what we had to prove

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